THE KCM ANNUAL CONFERENCE 2015

ENGAGE!

THE HILTON HOTEL

LEXINGTON, KY

MARCH 9-10, 2015

TITLE OF PRESENTATION: IDENTIFYING STUDENT MISCONCEPTIONS AND HOW TO ALLEVIATE THEM IN THE COMMON CORE ENVIRONMENT

DATE AND TIME: MONDAY, MARCH 9, 2015, 2:40 P.M. -3:50 P.M.

LOCATION: CRIMSON CLOVER

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IDENTIFYING STUDENT MISCONCEPTIONS AND HOW TO ALLEVIATE THEM IN THE COMMON CORE ENVIRONMENT

JAY L. SCHIFFMAN

Abstract: **Mathematics educators are aware of misconceptions like the "freshman dream" when squaring a binomial** $\left((a+b)^2 = a^2 + b^2\right)$, $2^3 \cdot 3^4 = 6^{12}$, $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$, $\frac{24}{48}=\frac{2}{8}$ **or** $\sin 2 \cdot x = 2 \cdot \sin x$ **permeating at both the secondary and undergraduate levels to name just a few. This workshop will engage participants to alleviate such persistent difficulties via the use of manipulatives, technology and counterexamples to strengthen conceptual understanding.**

SOME PROBLEMS AND DISCUSSION ACTIVITIES IN IDENTIFYING STUDENT MISCONCEPTIONS AND HOW TO ALLEVIATE THEM IN THE COMMON CORE ENVIRONMENT:

I. Consider the 'freshman dream" when squaring the binomial $(a+b)^2 = a^2 + b^2$. Think of several avenues one can partake of to alleviate this dilemma. Your list should include area models, the binomial theorem, counterexamples and technology. Extend the problem to higher powers of $a+b$.

II. Correct the error in the following problem involving addition of fractions: $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$. Why does the original answer not make sense? Using cuisenaire rods and diagrams, obtain a correct solution which supports the standard algorithm.

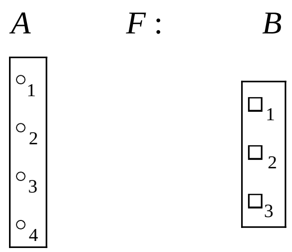
III. Correct the false statement $\frac{24}{48}=\frac{2}{8}$. Give several examples where crossing out the middle two digits (the second digit in the numerator and the first digit in the denominator) actually yields the correct simplification, but for the wrong reason. We are looking for anomalies. What is the correct procedure for simplifying fractions such as those we are encountering using manipulatives and technology?

IV. Correct the false statement $2^3 \cdot 3^4 = 6^{12}$. Why is there no gimmick or rule in this case? Discuss ideas including number sense in considering the relative size of the two sides of this "equation."

V. Many students erroneously believe that $\sin(2\cdot x) = 2\cdot \sin x$. Correct this false statement and furnish several strategies one can use to demonstrate that the original statement is incorrect. Is there any time when the original statement is actually true?

VI. Two students assert that the domain of the function $f(x)=\sqrt{x-3}$ is respectively given by $\{4,5,6,7,...\}$ *and* $\{3,4,5,6,7,...\}$. Explain how you would help these students to obtain a better foothold on the concept of domain. How would the answer change if one were speaking of the sequence s_n defined by $s_n=\sqrt{n-3}$.

VII. A student asserts that the following arrowed diagram represents a function from the set A into the set B if the student constructs arrows as follows: $\circ_1 \to \square_1$, $\circ_2 \to \square_2$, $\circ_3 \to \square_3$.



What is the student missing in his/her analysis?

VIII. A teacher in algebra tells his students that the following property of radicals is always true: $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$. **A student then notices the following demonstration in the textbook that shows** $-1 = 1$: $-1 = i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1 \cdot -1} = \sqrt{1} = 1$. **The teacher is stunned! He/she has neglected to consider an exception. What is the exception?**

IX. Consider the following improper use of a well-known algebraic property involving quadratic equations:

$$
\begin{aligned}
&(6 - x) \cdot (x - 9) = -4 \\
&(6 - x = -4) \vee (x - 9 = -4) \\
&(-x = -10) \vee (x = 5) \\
&(x = 10) \vee (x = 5)
\end{aligned}
$$

The student then checks the solutions:

$$
\begin{aligned}
x = 5: \quad & (6 - x) \cdot (x - 9) = -4 \\
& (6 - 5) \cdot (5 - 9) = -4 \\
& 1 \cdot -4 \qquad = -4 \\
& -4 \qquad = -4 \\
x = 10: \quad & (6 - x) \cdot (x - 9) = -4 \\
& (6 - 10) \cdot (10 - 9) = -4 \\
& -4 \cdot 1 \qquad = -4 \\
& -4 \qquad = -4
\end{aligned}
$$

Both solutions check! Is this a shortcut to the traditional method?

What is wrong with the method?

Show that the method works on the following quadratic equations:

$(7 - x) \cdot (x - 9) = -3$ *and* $(8 - x) \cdot (x - 9) = -2$.

Are there any others in the family that work via this erroneous method?

Solve each of these problems correctly.

Reconcile the above situation with the quadratic equation $x \cdot (x-5) = -6.$

X. Consider the trigonometric equation $\cos^2\theta - \cos\theta = 0;\ 0 \le \theta < 2\cdot\pi.$ **This trigonometric equation is of the second degree. Hence there can be no more than two roots. Is this correct? Explain.**

XI. Many students confuse number properties such as commutativity and associativity. On the set of real numbers, can one find a binary operation that is (a). commutative and associative; (b). neither commutative nor associative; (c). associative but not commutative; (d). commutative but not associative. Hence the properties of commutativity and associativity are logically independent in the sense that neither one implies the other. Try to furnish illustrations of the four possible cases.

XII. A student remarked that he found a prime number generating formula in the sense that all the output values are prime numbers: $p(n) = n^2 - 7\cdot n + 53;\ n \in W = \{0,1,2,3,4,5,\ldots\}.$ **By going far enough out in the sequence, prove him wrong! Here we must persevere in problem solving and use appropriate tools strategically.**

SOLUTIONS TO SOME PROBLEMS AND DISCUSSION ACTIVITIES IN IDENTIFYING STUDENT MISCONCEPTIONS AND HOW TO ALLEVIATE THEM IN THE COMMON CORE ENVIRONMENT:

I. The easiest way to show that $(a+b)^2 \neq a^2+b^2$ is via a counterexample. Let $a=2$ *and* $b=3$. Then $(a+b)^2=(2+3)^2=5^2=25$ *and* $a^2+b^2=2^2+3^2=4+9=13$. Hence $(a+b)^2 \neq a^2+b^2$.

A second way of viewing this is via an area model in which one takes a rectangle of length $a+b$ and width $a+b$. Subdivide the length of the rectangle $a+b$ into lengths a *and* b respectively. The width of the rectangle $a+b$ can similarly be subdivided into lengths a *and* b as well. This gives us four rectangles which partition the original square as seen below:

	A	B
A	$A*A$	$A*B$
B	$B*A$	$B*B$

The sums of the partial products (which represent the sums of the areas of the rectangles) is $A*A+A*B+B*A+B*B=A*A+A*B+A*B+B*B=A^2+(A*B+A*B)+B^2=A^2+2*A*B+B^2=(A+B)^2$. Note that

$(2+3)^2=(2+3)\cdot(2+3)=2\cdot(2+3)+3\cdot(2+3)=2\cdot2+2\cdot3+3\cdot2+3\cdot3=4+6+6+9=25=5^2$.

Hence $(2+3)^2=2^2+2\cdot2\cdot3+3^2=4+12+9=25=5^2$. It is important to note that if $a=0$ *or* $b=0$ (or both), then $(a+b)^2=a^2+b^2$. Using technology such as a calculator equipped with CAS (a Computer Algebra System), we view the following in **FIGURES 1-2** using the Expand command:

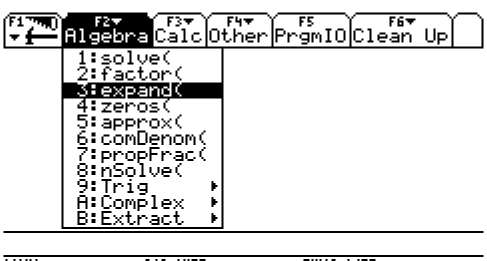


FIGURE 1

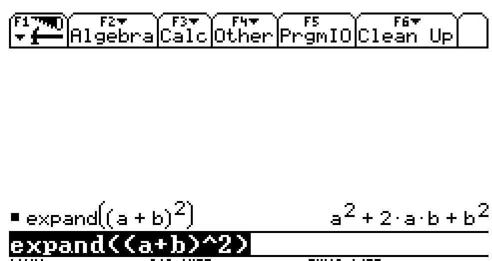


FIGURE 2

Let us examine higher powers of $a+b$. A configuration known as Pascal's Triangle aids in achieving our desired goal. Here are the first five rows of Pascal's Triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Based on the initial five rows of Pascal's Triangle, the next five rows are displayed. Let us write Rows 0-10 as follows:

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	**5**	10	10	5	1					
1	**6**	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	120	252	210	120	45	10	1

The binomial theorem asserts the following for whole numbers n with Pascal's Triangle furnishing the binomial coefficients:

$$(a+b)^n = C(n,0)\cdot a^n \cdot b^0 + C(n,1)\cdot a^{n-1}\cdot b^1 + C(n,2)\cdot a^{n-2}\cdot b^2 + C(n,3)\cdot a^{n-3}\cdot b^3 + \ldots + C(n,n)\cdot a^{n-n}\cdot b^n.$$

Here $C(n,k) = \dfrac{n!}{k!\cdot(n-k)!}$.

Thus we have the following results for the first eleven whole number ten powers of $a+b$:

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2\cdot a\cdot b + b^2$$
$$(a+b)^3 = a^3 + 3\cdot a^2\cdot b + 3\cdot a\cdot b^2 + b^3$$
$$(a+b)^4 = a^4 + 4\cdot a^3\cdot b + 6\cdot a^2\cdot b^2 + 4\cdot a\cdot b^3 + b^4$$
$$(a+b)^5 = a^5 + 5\cdot a^4\cdot b + 10\cdot a^3\cdot b^2 + 10\cdot a^2\cdot b^3 + 5\cdot a\cdot b^4 + b^5$$
$$(a+b)^6 = a^6 + 6\cdot a^5\cdot b + 15\cdot a^4\cdot b^2 + 20\cdot a^3\cdot b^3 + 15\cdot a^2\cdot b^4 + 6\cdot a\cdot b^5 + b^6$$
$$(a+b)^7 = a^7 + 7\cdot a^6\cdot b + 21\cdot a^5\cdot b^2 + 35\cdot a^4\cdot b^3 + 35\cdot a^3\cdot b^4 + 21\cdot a^2\cdot b^5 + 7\cdot a\cdot b^6 + b^7$$
$$(a+b)^8 = a^8 + 8\cdot a^7\cdot b + 28\cdot a^6\cdot b^2 + 56\cdot a^5\cdot b^3 + 70\cdot a^4\cdot b^4 + 56\cdot a^3\cdot b^5 + 28\cdot a^2\cdot b^6 + 8\cdot a\cdot b^7 + b^8$$
$$(a+b)^9 = a^9 + 9\cdot a^8\cdot b + 36\cdot a^7\cdot b^2 + 84\cdot a^6\cdot b^3 + 126\cdot a^5\cdot b^4 + 126\cdot a^4\cdot b^5 + 84\cdot a^3\cdot b^6 + 36\cdot a^2\cdot b^7 + 9\cdot a\cdot b^8 + b^9$$
$$(a+b)^{10} = a^{10} + 10\cdot a^9\cdot b + 45\cdot a^8\cdot b^2 + 120\cdot a^7\cdot b^3 + 210\cdot a^6\cdot b^4 + 252\cdot a^5\cdot b^5 + 210\cdot a^4\cdot b^6 + 120\cdot a^3\cdot b^7 + 45\cdot a^2\cdot b^8 + 10\cdot a\cdot b^9 + b^{10}$$

Technology can offer additional morsels. Let us secure the initial eleven whole number powers of $a+b$ with the aid of the TI-89 in **FIGURES 3-8:**

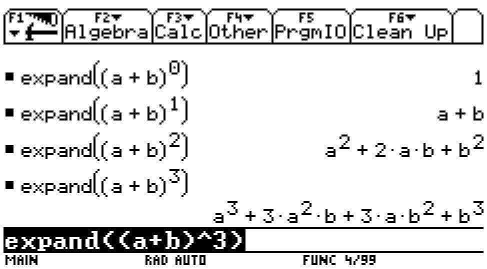


FIGURE 3

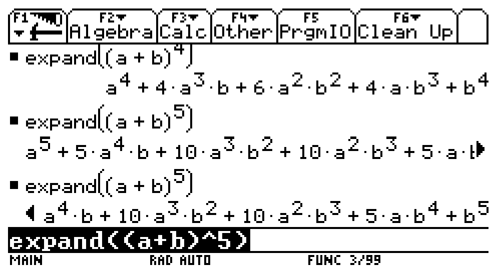


FIGURE 4

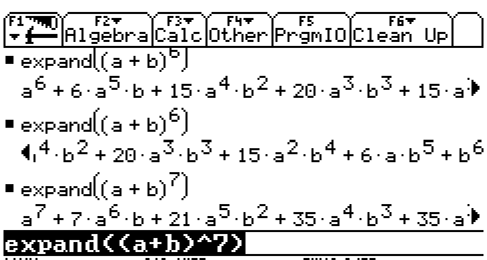


FIGURE 5

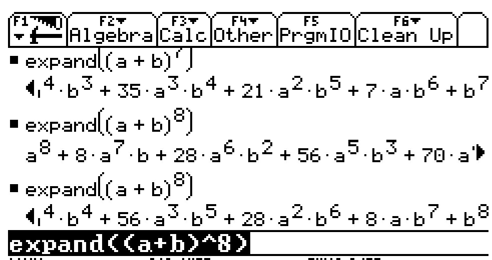


FIGURE 6

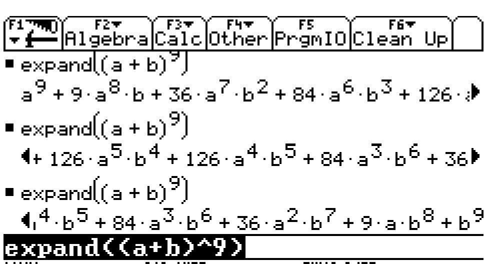


FIGURE 7

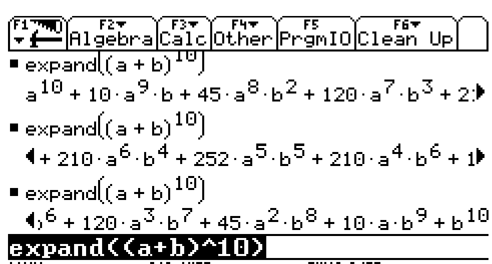


FIGURE 8

Using the TI-89, one can also obtain the entries in any row of Pascal's Triangle. See **FIGURES 9-10** for the set up and the entries in the first ten rows of the triangle in **FIGURES 11-13.**

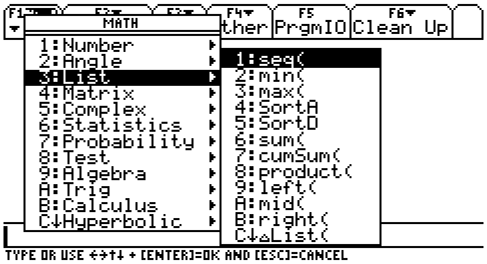


FIGURE 9

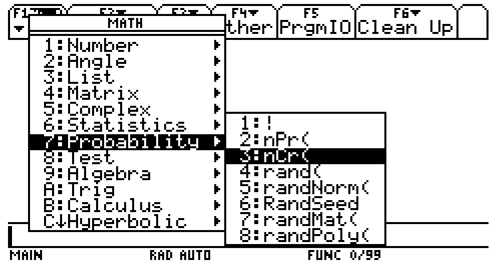


FIGURE 10

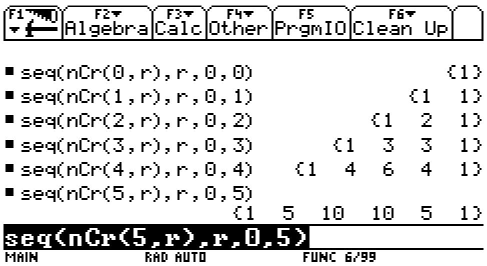


FIGURE 11

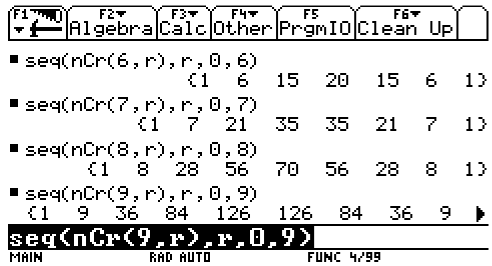


FIGURE 12

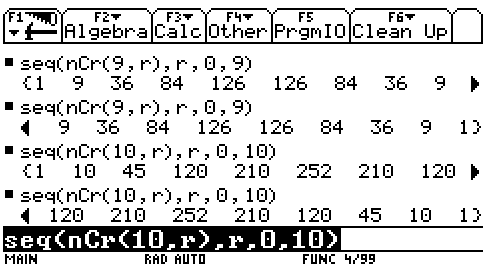


FIGURE 13

II. We correct the error in the following problem involving addition of fractions: $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$ and explain why the original answer is not meaningful and using cuisenaire rods and diagrams obtain a correct solution which supports the standard algorithm.

The standard method to add the fractions $\frac{1}{2}$ *and* $\frac{1}{4}$ is to secure a common denominator. The least common denominator in this case is 4. It is important for both the student and teacher to realize that finding the LCD for a set of fractions is considering the LCM (least Common Multiple) of the involved denominators of the fractions. Traditionally using the standard algorithm, one obtains $\frac{1}{2}+\frac{1}{4}=\frac{1}{2}\cdot\frac{2}{2}+\frac{1}{4}=\frac{1\cdot 2}{2\cdot 2}+\frac{1}{4}=\frac{2}{4}+\frac{1}{4}=\frac{2+1}{4}=\frac{3}{4}$. The computation demonstrates procedural fluency without necessarily an understanding of why $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$. The original answer given as $\frac{2}{6}$ is untenable since the first addend $\frac{1}{2}$ in the problem is already larger than the final sum $\frac{2}{6} = \frac{1}{3}$. The idea of having good number sense is paramount to understanding the underpinnings of elementary mathematics.

When using cuisenaire rods and other manipulatives such as fraction bars, one must be able to identify the unit in the problem. Cuisenaire rods are colored rods with each rod assigned a specific integer value between on and ten. For example, White = 1, Red = 2, Light Green = 3, Purple = 4, Yellow = 5, Dark Green = 6, Black = 7, Brown = 8, Blue = 9, Orange = 10. Hence we can use the Cuisenaire rods to discover equivalent fractions or to use strip or tape diagrams to perform operations with fractions.

III. $\frac{24}{48} = \frac{2\cdot 2\cdot 2\cdot 3}{2\cdot 2\cdot 2\cdot 2\cdot 3} = \frac{1}{2}$. When simplifying fractions, the idea is to divide both numerator and denominator by the greatest common divisor of both which is 24. There are many models one can use to secure the GCD (Greatest Common Divisor) for a set of two positive integers. One way to form a systematic list which enumerates the divisors of each integer, search for the common divisors and secure the largest of these.

In our example, the divisors of 24 and 48 can be secured in the following table:

Divisors of 24: 1, 2, 3, 4, 6, 8, 12, 24
Divisors of 48: 1, 2, 3, 4, 6, 8, 12, 24, 48

We next color the common divisors of 24 and 48 in Red and the Greatest of these common divisors (24) in Green as shown below:

Divisors of 24: 1, 2, 3, 4, 6, 8, 12, 24
Divisors of 48: 1, 2, 3, 4, 6, 8, 12, 24, 48

Divisors of 24: 1, 2, 3, 4, 6, 8, 12, 24
Divisors of 48: 1, 2, 3, 4, 6, 8, 12, 24, 48

Other models one can consider are the intersection of sets method, the Euclidean algorithm and prime factorization. The graphing calculator can aid as well. See **FIGURES 14-15:**

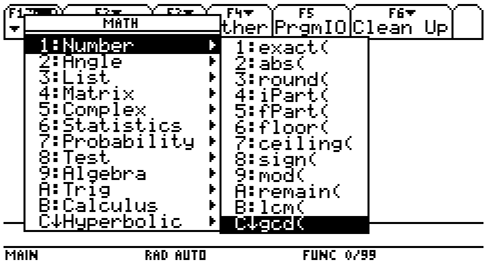


FIGURE 14

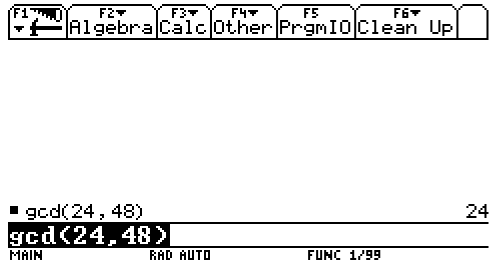


FIGURE 15

There are three anomalies where crossing out the middle two digits actually yields the correct answer: $\frac{16}{64}=\frac{1}{4}, \frac{19}{95}=\frac{1}{5}$ *and* $\frac{26}{65}=\frac{2}{5}.$ Of course, these are the exceptions to the rule.

In all the other possible combinations of fractions that can actually be simplified, this homemade rule does not work!

IV. The equation $2^3 \cdot 3^4 = 6^{12}$ could not possibly be correct without even computing. The right hand side of the equation is vastly larger than the left hand side. Note that $6^{12} = (2 \cdot 3)^{12} = 2^{12} \cdot 3^{12}.$ One has twelve factors of six which correlates to the product of twelve factors of two with twelve factors of three. The left hand side represents the product of three factors of two with four factors of three. In this problem, what is more important is the proper use of number sense rather than merely the correct answer. The correct answer can be secured via the calculator as in **FIGURE 16** where we also compare the relative size of the two quantities $2^3 \cdot 3^4$ and 6^{12}:

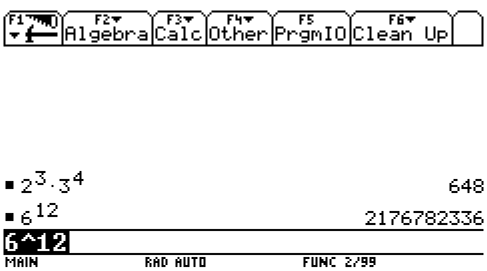


FIGURE 16

V. A substantial knowledge of the trigonometric functions is vital for all students pursuing stem careers. Unfortunately, this understanding is not always prevalent in calculus courses which feed on trigonometry. For example, many students erroneously believe that $\sin(2 \cdot x) = 2 \cdot \sin x.$ We correct this false statement and furnish several strategies one can use to demonstrate that the original statement is incorrect and discover that there again are anomalies when this original statement is actually true. Let us first note the use of the counterexample to falsify the equation. Consider $x = \frac{\pi}{4}.$ Then $\sin(2 \cdot x) = \sin\left(2 \cdot \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \neq \sqrt{2} = 2 \cdot \frac{\sqrt{2}}{2} = 2 \cdot \sin\frac{\pi}{4} = 2 \cdot \sin x.$

Another method of viewing this pitfall is to graph both functions with the aid of a graphing calculator handheld and view the graphs as well as a table as in **FIGURES 17-28:**

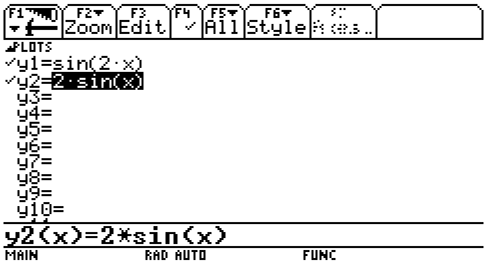


FIGURE 17

FIGURE 18

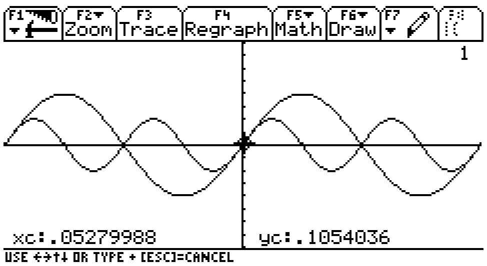


FIGURE 19

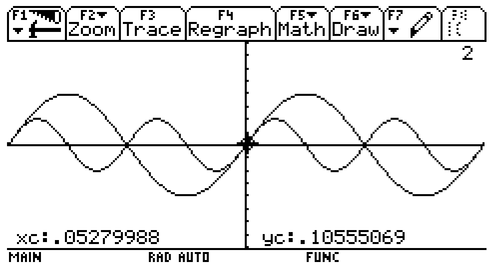


FIGURE 20

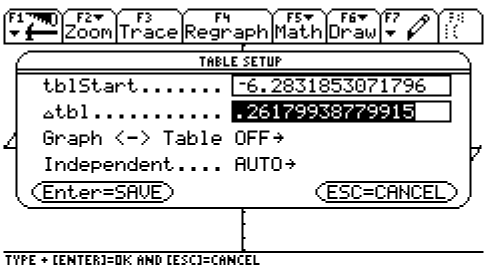


FIGURE 21

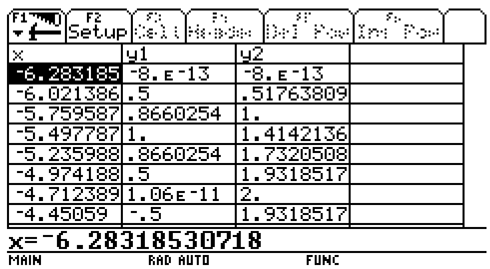


FIGURE 22

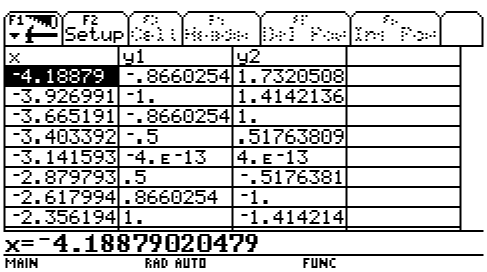


FIGURE 23

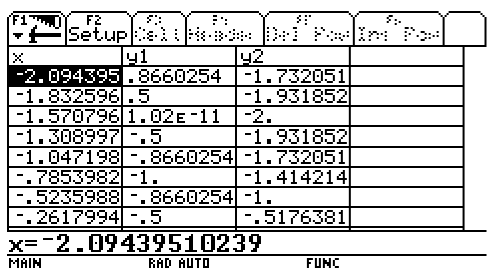
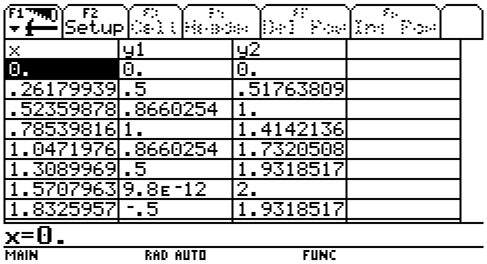
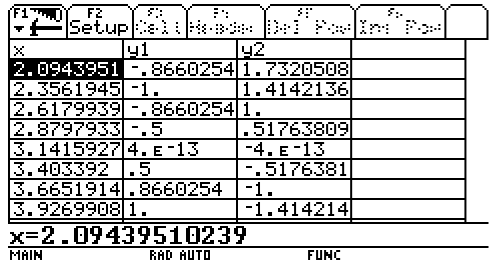


FIGURE 24



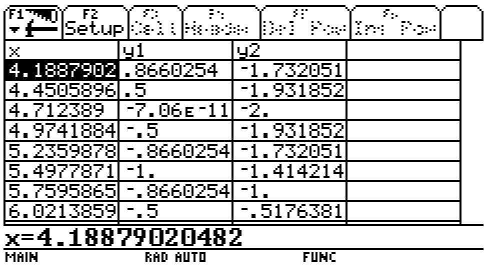
x	y1	y2
0.	0.	0.
.26179939	.5	.51763809
.52359878	.8660254	1.
.78539816	1.	1.4142136
1.0471976	.8660254	1.7320508
1.3089969	.5	1.9318517
1.5707963	9.8E-12	2.
1.8325957	-.5	1.9318517

FIGURE 25



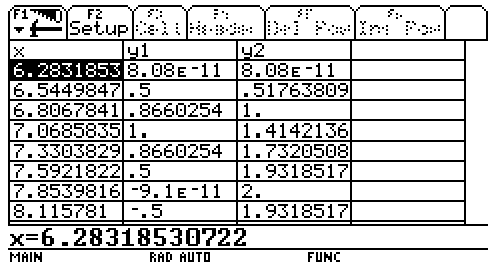
x	y1	y2
2.0943951	-.8660254	1.7320508
2.3561945	-1.	1.4142136
2.6179939	-.8660254	1.
2.8797933	-.5	.51763809
3.1415927	4.E-13	-4.E-13
3.403392	.5	-.5176381
3.6651914	.8660254	-1.
3.9269908	1.	-1.414214

FIGURE 26



x	y1	y2
4.1887902	.8660254	-1.732051
4.4505896	.5	-1.931852
4.712389	-7.06E-11	-2.
4.9741884	-.5	-1.931852
5.2359878	-.8660254	-1.732051
5.4977871	-1.	-1.414214
5.7595865	-.8660254	-1.
6.0213859	-.5	-.5176381

FIGURE 27



x	y1	y2
6.2831853	8.08E-11	8.08E-11
6.5449847	.5	.51763809
6.8067841	.8660254	1.
7.0685835	1.	1.4142136
7.3303829	.8660254	1.7320508
7.5921822	.5	1.9318517
7.8539816	-9.1E-11	2.
8.115781	-.5	1.9318517

FIGURE 28

On the other hand, if $x = k \cdot \pi$ where $k \in \mathbb{Z}$, then it is indeed the case that $\sin(2 \cdot x) = 2 \cdot \sin x.$ Note this holds since $\sin(2 \cdot k \cdot \pi) = 0 = 2 \cdot \sin(k \cdot \pi).$

Next we show that $\sin(2 \cdot x) = 2 \cdot \sin x \cdot \cos x.$ Assuming that the sum formula for the sine function has been covered, namely
$\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y.$ *If* $y = x,$ *then* $\sin(2 \cdot x) = \sin(x+x) =$
$\sin x \cdot \cos x + \cos x \cdot \sin x = \sin x \cdot \cos x + \sin x \cdot \cos x = 2 \cdot \sin x \cdot \cos x.$

Using the calculator can lend credence that the identity might be true, but the above is a formal proof. See **FIGURES 29-39:**

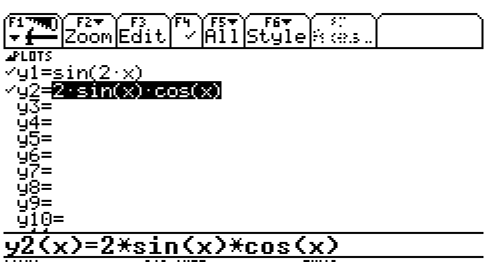


FIGURE 29

FIGURE 30

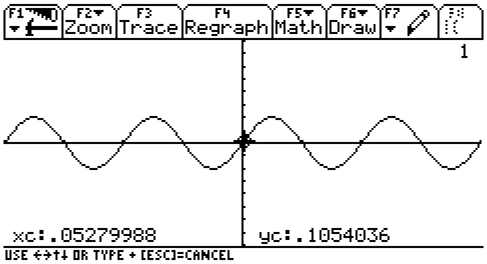


FIGURE 31

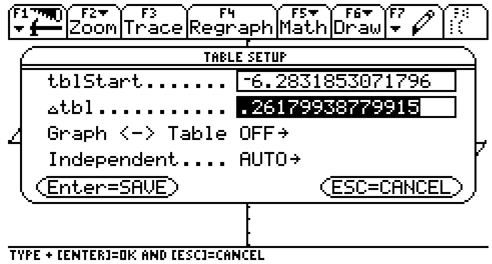
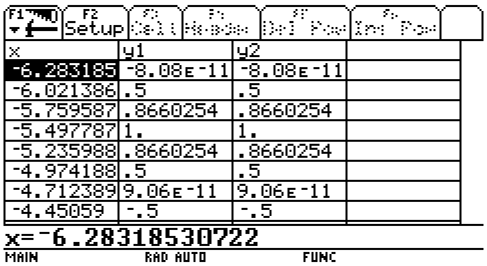


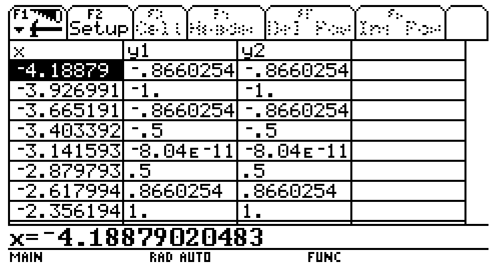
FIGURE 32



x	y1	y2
-6.283185	-8.08E-11	-8.08E-11
-6.021386	.5	.5
-5.759587	.8660254	.8660254
-5.497787	1.	1.
-5.235988	.8660254	.8660254
-4.974188	.5	.5
-4.712389	9.06E-11	9.06E-11
-4.45059	-.5	-.5

x=-6.28318530722
MAIN RAD AUTO FUNC

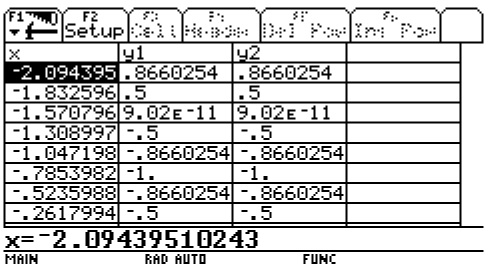
FIGURE 33



x	y1	y2
-4.18879	-.8660254	-.8660254
-3.926991	-1.	-1.
-3.665191	-.8660254	-.8660254
-3.403392	-.5	-.5
-3.141593	-8.04E-11	-8.04E-11
-2.879793	.5	.5
-2.617994	.8660254	.8660254
-2.356194	1.	1.

x=-4.18879020483
MAIN RAD AUTO FUNC

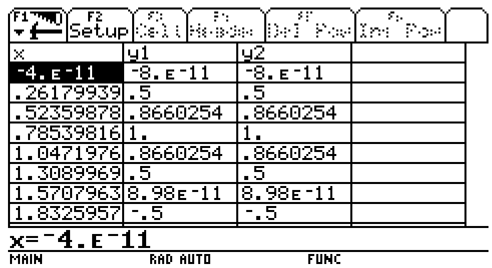
FIGURE 34



x	y1	y2
-2.094395	.8660254	.8660254
-1.832596	.5	.5
-1.570796	9.02E-11	9.02E-11
-1.308997	-.5	-.5
-1.047198	-.8660254	-.8660254
-.7853982	-1.	-1.
-.5235988	-.8660254	-.8660254
-.2617994	-.5	-.5

x=-2.09439510243
MAIN RAD AUTO FUNC

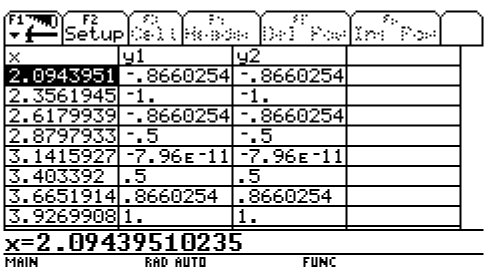
FIGURE 35



x	y1	y2
-4.E-11	-8.E-11	-8.E-11
.26179939	.5	.5
.52359878	.8660254	.8660254
.78539816	1.	1.
1.0471976	.8660254	.8660254
1.3089969	.5	.5
1.5707963	8.98E-11	8.98E-11
1.8325957	-.5	-.5

x=-4.E-11
MAIN RAD AUTO FUNC

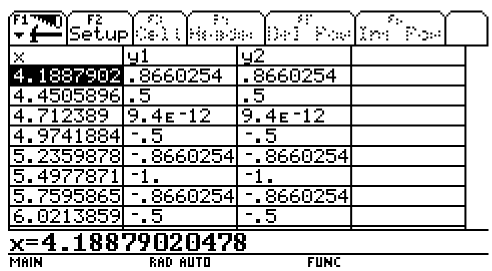
FIGURE 36



x	y1	y2
2.0943951	-.8660254	-.8660254
2.3561945	-1.	-1.
2.6179939	-.8660254	-.8660254
2.8797933	-.5	-.5
3.1415927	-7.96E-11	-7.96E-11
3.403392	.5	.5
3.6651914	.8660254	.8660254
3.9269908	1.	1.

x=2.09439510235
MAIN RAD AUTO FUNC

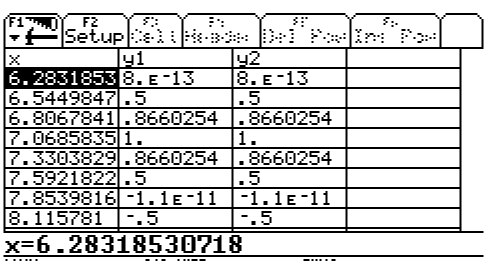
FIGURE 37



x	y1	y2
4.1887902	.8660254	.8660254
4.4505896	.5	.5
4.712389	9.4E-12	9.4E-12
4.9741884	-.5	-.5
5.2359878	-.8660254	-.8660254
5.4977871	-1.	-1.
5.7595865	-.8660254	-.8660254
6.0213859	-.5	-.5

x=4.18879020478
MAIN RAD AUTO FUNC

FIGURE 38



x	y1	y2
6.2831853	8.E-13	8.E-13
6.5449847	.5	.5
6.8067841	.8660254	.8660254
7.0685835	1.	1.
7.3303829	.8660254	.8660254
7.5921822	.5	.5
7.8539816	-1.1E-11	-1.1E-11
8.115781	-.5	-.5

x=6.28318530718
MAIN RAD AUTO FUNC

FIGURE 39

VI. Two students respectively assert that the domain of the function $f(x)=\sqrt{x-3}$ is $\{4,5,6,7,...\}$ *and* $\{3,4,5,6,7,...\}$. It is essential to explain how one can help these students obtain a better foothold on the concept of domain and further discuss how the answer would change if one were speaking of the sequence s_n defined by $s_n=\sqrt{n-3}$. The major difficulty I find is that the students are thinking in terms of the set of integers, a discrete set, in contrast to the continuous set of real numbers. Consider the function $f(x)=\sqrt{x-3}$. The Domain of $f=\{x\in\mathbb{R}\mid x-3\geq 0\}=\{x\in\mathbb{R}\mid x\geq 3\}$. We are asserting that the domain consists of all real numbers at least three. This includes all rational and irrational numbers such as $3.\overline{3}=3\frac{1}{3}=\frac{10}{3}$, $\sqrt{10}\approx 3.16227766017$, 3.5 *and* 23. All of these real numbers with the exception of $\sqrt{10}$ are rational numbers. Let us form the graph as well as a table for the function in question as depicted in **FIGURES 40-45:**

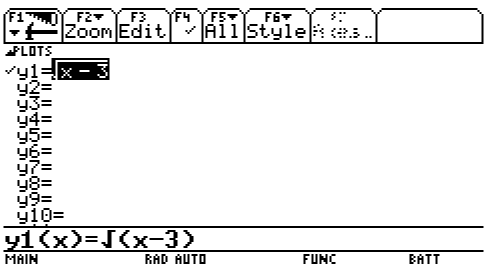


FIGURE 40

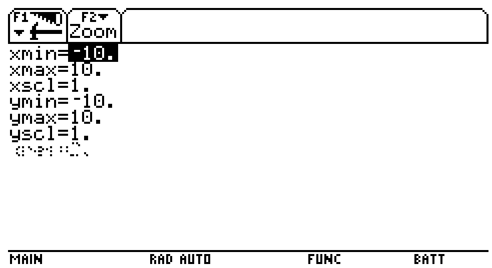


FIGURE 41

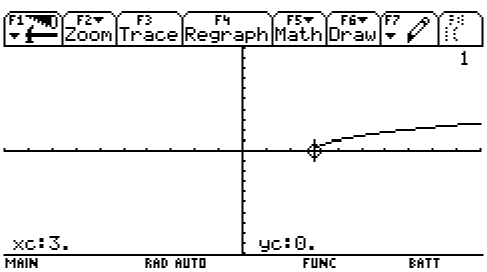


FIGURE 42

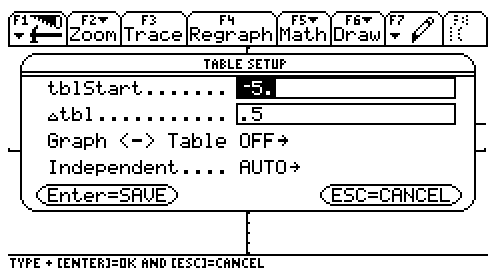


FIGURE 43

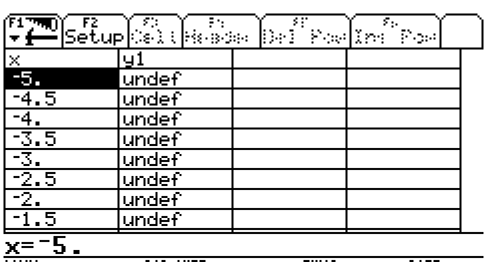


FIGURE 44

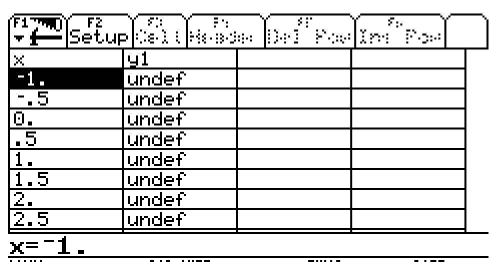
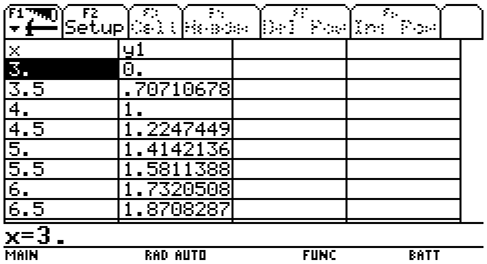
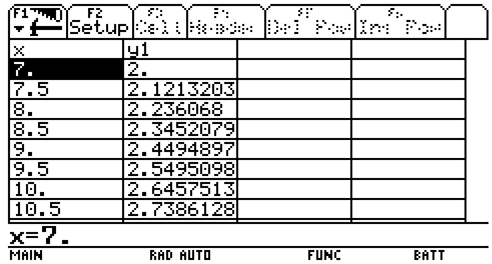


FIGURE 45



x	y1
3.	0.
3.5	.70710678
4.	1.
4.5	1.2247449
5.	1.4142136
5.5	1.5811388
6.	1.7320508
6.5	1.8708287

FIGURE 46



x	y1
7.	2.
7.5	2.1213203
8.	2.236068
8.5	2.3452079
9.	2.4494897
9.5	2.5495098
10.	2.6457513
10.5	2.7386128

FIGURE 47

From the above analysis, it is clear that the domain contains more than integers that exceed 3. In fact, let us obtain a value when $x = 3 \cdot \sqrt{2}$. See **FIGURES 48-49:**

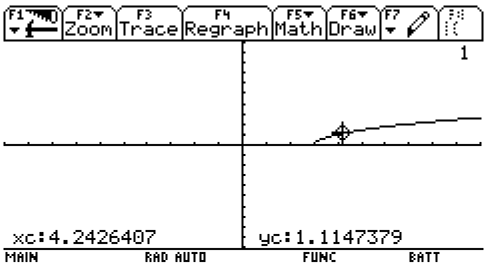


FIGURE 48

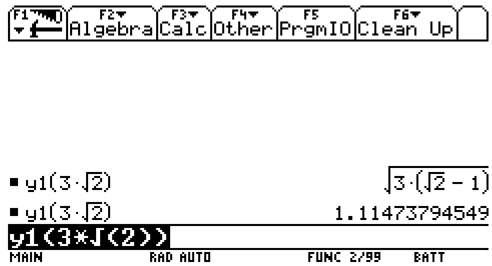


FIGURE 49

For the sequence $s_n = \sqrt{n-3}$, we consider only positive integer inputs. By definition, a sequence is a function whose domain is the set of positive integers. See **FIGURES 50-54.** Make sure the MODE is SEQUENCE, not FUNCTION.

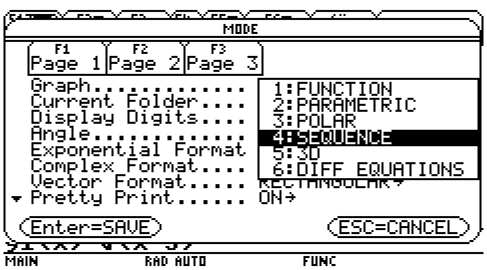


FIGURE 50

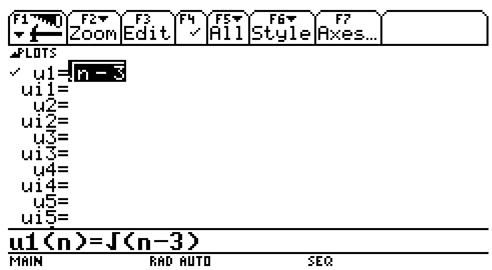


FIGURE 51

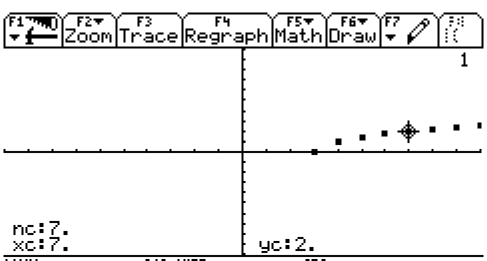


FIGURE 52

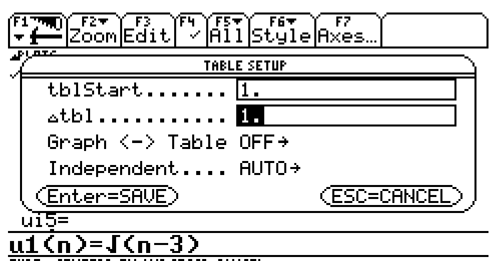
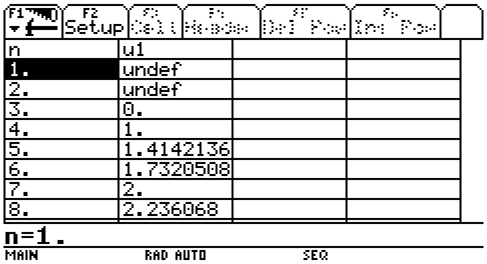


FIGURE 53

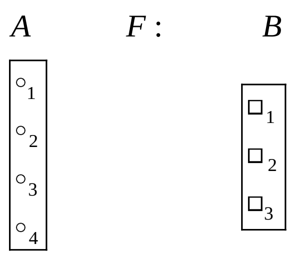


n	u1		
1.	undef		
2.	undef		
3.	0.		
4.	1.		
5.	1.4142136		
6.	1.7320508		
7.	2.		
8.	2.236068		

FIGURE 54

Note that in sequence graphing, the points are discrete or isolated (separated) in contrast to function graphing where all the points are connected.

VII. A student asserts that the following arrowed diagram represents a function from the set A into the set B if the student constructs arrows as follows: $\circ_1 \to \square_1, \circ_2 \to \square_2, \circ_3 \to \square_3$.



The student is missing the fact that in order for an arrowed diagram model to constitute a function, each element in the domain A must map to one and only one element in the codomain B. This is not the case since the element $\circ_4 \in A$ does not map to any element in B.

VIII. A teacher in algebra tells his students that the following property of radicals is always true: $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$. A student then notices the following demonstration in the textbook that shows $-1 = 1$: $-1 = i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1 \cdot -1} = \sqrt{1} = 1$. The teacher is stunned! He/she has neglected to consider an exception. The problem lies in the step

$\sqrt{(-1) \cdot (-1)} = \sqrt{1} = 1$ *while* $\sqrt{-1} \cdot \sqrt{-1} = i \cdot i = i^2 = -1$. The rule that

$$f(x) = 2 \cdot x + 1 \text{ and } g(x) = \frac{x-1}{2}.$$

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x-1}{2}\right) = 2 \cdot \left(\frac{x-1}{2} + 1\right) = x - 1 + 1 = x \text{ and}$$

$$(g \circ f)(x) = g(f(x)) = g(2 \cdot x + 1) = \left(\frac{2 \cdot x + 1 - 1}{2}\right) = \frac{2 \cdot x}{2} = x.$$

$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ works if at least one of the factors is non-negative which is not the case; for both factors are equal to -1. This is the exception to this property of radicals.

IX. Consider the following improper use of a well-known algebraic property involving quadratic equations:

$$(6-x)\cdot(x-9)=-4$$
$$(6-x=-4)\vee(x-9=-4)$$
$$(-x=-10)\vee(x=5)$$
$$(x=10)\vee(x=5)$$

The student then checks the solutions:

$$\begin{aligned} x=5:\quad &(6-x)\cdot(x-9)=-4\\ &(6-5)\cdot(5-9)=-4\\ &1\cdot -4 \qquad\quad =-4\\ &\;\;-4 \qquad\quad\;\; =-4\\ x=10:\quad &(6-x)\cdot(x-9)=-4\\ &(6-10)\cdot(10-9)=-4\\ &-4\cdot 1 \qquad\quad =-4\\ &\;\;-4 \qquad\quad\;\; =-4 \end{aligned}$$

Both solutions check! Is this a shortcut to the traditional method?

What is wrong with the method?

Show that the method works on the following quadratic equations:

$(7-x)\cdot(x-9)=-3$ *and* $(8-x)\cdot(x-9)=-2.$

Are there any others in the family that work via this erroneous method?

Solve each of these problems correctly.

Reconcile the above situation with the quadratic equation $x\cdot(x-5)=-6.$

One here is applying the zero factors property incorrectly. It is indeed the case that if
$If\ a,b\in\mathbb{R}\ s.t.\ a\cdot b=0,\ then\ (a=0)\vee(b=0).$ This is easily proven. For suppose that
$a\cdot b=0\ but\ a\neq 0.$ We show $b=0.$
$a\neq 0\Rightarrow \frac{1}{a}\ exists.\ \frac{1}{a}\cdot 0=\frac{1}{a}\cdot(a\cdot b)=\left(\frac{1}{a}\cdot a\right)\cdot b=1\cdot b=b\Rightarrow b=0.$ If one applies this false narrative to these problems where the right hand side is not zero, the gimmick actually works. We consider $(7-x)\cdot(x-9)=-3$ *and* $(8-x)\cdot(x-9)=-2.$ Observe the following:

$$(7-x)\cdot(x-9)=-3$$
$$(7-x=-3)\vee(x-9=-3)$$
$$(-x=-10)\vee(x=6)$$
$$(x=10)\vee(x=6)$$

The check is provided below:

$$\begin{aligned} x=6: \quad & (7-x)\cdot(x-9)=-3 \\ & (7-6)\cdot(6-9)=-3 \\ & 1\cdot -3 \qquad\quad =-3 \\ & \ \ -3 \qquad\quad\ \ =-3 \\ x=10: \quad & (7-x)\cdot(x-9)=-3 \\ & (7-10)\cdot(10-9)=-3 \\ & -3\cdot 1 \qquad\quad =-3 \\ & \ \ -3 \qquad\quad\ \ =-3 \end{aligned}$$

$$(8-x)\cdot(x-9)=-2$$
$$(8-x=-2)\vee(x-9=-2)$$
$$(-x=-10)\vee(x=7)$$
$$(x=10)\vee(x=7)$$

The check is provided here:

$$\begin{aligned} x=7: \quad & (8-x)\cdot(x-9)=-2 \\ & (8-7)\cdot(7-9)=-2 \\ & 1\cdot -2 \qquad\quad =-2 \\ & \ \ -2 \qquad\quad\ \ =-2 \\ x=10: \quad & (8-x)\cdot(x-9)=-2 \\ & (8-10)\cdot(10-9)=-2 \\ & -2\cdot 1 \qquad\quad =-2 \\ & \ \ -2 \qquad\quad\ \ =-2 \end{aligned}$$

It is amazing that this gimmick actually works for the family $(k-x)\cdot(x-9)=k-10,\ k\in\mathbb{Z}$.

To see this, note that

$$\left(k-x\right)\cdot\left(x-9\right)=k-10$$
$$\left(k-x=k-10\right)\vee\left(x-9=k-10\right)$$
$$\left(-x=-10\right)\vee\left(x=k-10+9\right)$$
$$\left(x=10\right)\vee\left(x=k-1\right)$$

The check is furnished below:

$$\begin{aligned} x=k-1:\quad & \left(k-x\right)\cdot\left(x-9\right)=k-10 \\ & \left(k-\left(k-1\right)\right)\cdot\left(k-1-9\right)=k-10 \\ & \left(k-k+1\right)\cdot\left(k-10\right) \qquad =k-10 \\ & 1\cdot\left(k-10\right) \qquad\qquad =k-10 \\ & k-10 \qquad\qquad\qquad =k-10 \\ x=10:\quad & \left(k-x\right)\cdot\left(x-9\right)=k-10 \\ & \left(k-10\right)\cdot\left(10-9\right)=k-10 \\ & \left(k-10\right)\cdot 1 \qquad =k-10 \\ & k-10 \qquad\qquad =k-10 \end{aligned}$$

On the other hand, this gimmick does not work on the equation $x\left(x-5\right)=-6$; for $\left(x=-6\right)\vee\left(x-5=-6\right)\Leftrightarrow\left(x=-6\right)\vee\left(x=-1\right)$.

Check:

$$\begin{aligned} x=-6:\quad & x\cdot\left(x-5\right)=-6 \\ & -6\cdot\left(-6-5\right)=-6 \\ & -6\cdot-11 \qquad =-6 \\ & 66 \qquad\qquad =-6 \ (\textit{False}!) \\ x=-1:\quad & x\cdot\left(x-5\right)=-6 \\ & -1\cdot\left(-1-5\right)=-6 \\ & -1\cdot-6 \qquad =-6 \\ & 6 \qquad\qquad =-6 \ (\textit{False}!) \end{aligned}$$

The correct method for solving this problem is as follows:

$$x\cdot\left(x-5\right)=-6\Leftrightarrow x^{2}+5\cdot x+6=0\Leftrightarrow\left(x-2\right)\cdot\left(x-3\right)=0\Leftrightarrow\left(x-2=0\right)\vee\left(x-3=0\right)\Leftrightarrow\left(x=2\right)\vee\left(x=3\right).$$

To solve the problems $\left(7-x\right)\cdot\left(x-9\right)=-3$ *and* $\left(8-x\right)\cdot\left(x-9\right)=-2$ correctly, we of course, proceed as follows:

$(7-x)\cdot(x-9)=-3 \Leftrightarrow -x^2+16\cdot x-63=0 \Leftrightarrow x^2-16\cdot x+63=0 \Leftrightarrow (x-7)\cdot(x-9)=0 \Leftrightarrow (x-7=0)\vee(x-9=0) \Leftrightarrow (x=7)\vee(x=9).$
$(8-x)\cdot(x-9)=-2 \Leftrightarrow -x^2+17\cdot x-72=0 \Leftrightarrow x^2-17\cdot x+72=0 \Leftrightarrow (x-8)\cdot(x-9)=0 \Leftrightarrow (x-8=0)\vee(x-9=0) \Leftrightarrow (x=8)\vee(x=9).$

In order to correctly solve the family of equations of the form $(k-x)\cdot(x-9)=k-10,\ k\in\mathbb{Z},$ proceed as follows either by factoring or via the quadratic formula:

By factoring, we obtain

$$(k-x)\cdot(x-9)=k-10 \Leftrightarrow -x^2+k\cdot x+9\cdot x-9\cdot k=k-10 \Leftrightarrow x^2-k\cdot x-9\cdot x+9\cdot k=10-k \Leftrightarrow$$
$$x^2-(k+9)\cdot x+9\cdot k=10-k \Leftrightarrow x^2-(k+9)\cdot x+9\cdot k+k-10=0 \Leftrightarrow x^2-(k+9)\cdot x+(10\cdot k-10)=0 \Leftrightarrow$$
$$(x-10)\cdot(x-(k-1))=0 \Leftrightarrow (x-10=0)\vee(x-k+1=0) \Leftrightarrow (x=10)\vee(x=k-1).$$

If we employ the quadratic formula, we obtain from the quadratic equation
$x^2-(k+9)\cdot x+(10\cdot k-10)=0,$

$$x=\frac{-\left[-(k+9)\right]\pm\sqrt{\left[-(k+9)\right]^2-4\cdot 1\cdot(10\cdot k-10)}}{2\cdot 1}=\frac{(k+9)\pm\sqrt{(-1)^2\cdot(k+9)^2-4\cdot(10\cdot k-10)}}{2}=$$
$$\frac{(k+9)\pm\sqrt{(k^2+18\cdot k+81-40\cdot k+40)}}{2}=\frac{(k+9)\pm\sqrt{(k^2-22\cdot k+121)}}{2}=\frac{(k+9)\pm\sqrt{(k-11)^2}}{2}=$$
$$\left(\frac{(k+9)+\sqrt{(k-11)^2}}{2}\right)\vee\left(\frac{(k+9)-\sqrt{(k-11)^2}}{2}\right)=\left(\frac{k+9+k-11}{2}\right)\vee\left(\frac{k+9-(k-11)}{2}\right)=$$
$$\left(\frac{2\cdot k-2}{2}\right)\vee\left(\frac{k+9-k+11}{2}\right)=\frac{2\cdot(k-1)}{2}\vee\frac{20}{2}=(k-1)\vee 10.$$

X. Consider the trigonometric equation $\cos^2\theta-\cos\theta=0;\ 0\le\theta<2\cdot\pi.$ This trigonometric equation is of the second degree. Hence one might erroneously believe based on the study of polynomial theory that there can be no more than two roots. Is this correct? Explain. The conjecture is incorrect and is based on the theory of polynomial equations where any polynomial equation of degree two must have exactly two roots (not necessarily real and not necessarily distinct) by the n zeros theorem. Actually this trigonometric equation possesses THREE roots in the indicated interval. One can obtain this via multiple perspectives. Let us initially furnish an algebraic solution:

$$\cos^2\theta-\cos\theta=0 \Leftrightarrow \cos\theta\cdot(\cos\theta-1)=0 \Leftrightarrow (\cos\theta=0)\vee(\cos\theta-1=0) \Leftrightarrow$$
$$(\cos\theta=0)\vee(\cos\theta=1) \Leftrightarrow \left(\left(\theta=\frac{\pi}{2}\right)\vee\left(\theta=\frac{3\cdot\pi}{2}\right)\right)\vee(\theta=0)\in[0,2\cdot\pi).$$

A second method would be to graph the equations in TRIG MODE and use a graphing utility plus a TABLE FEATURE as well as the Solve Command. Please see **FIGURES 55-66:**

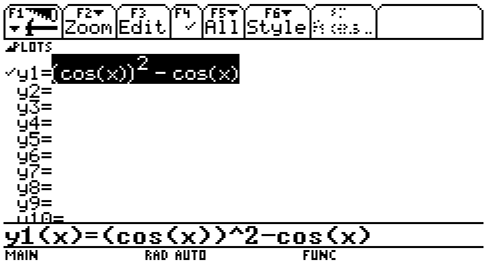


FIGURE 55

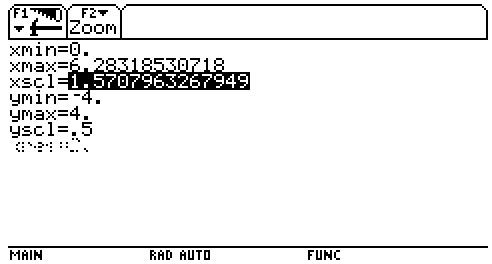


FIGURE 56

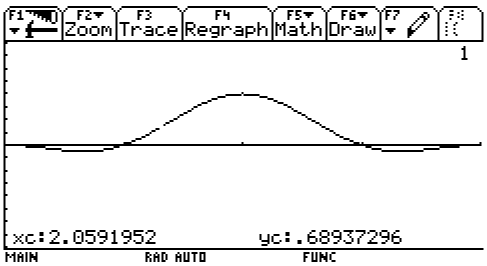


FIGURE 57

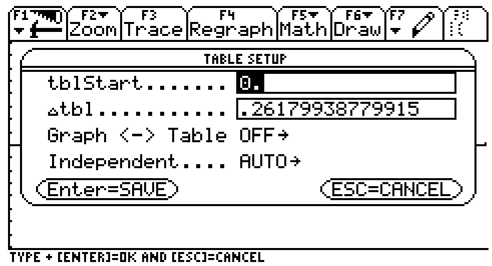


FIGURE 58

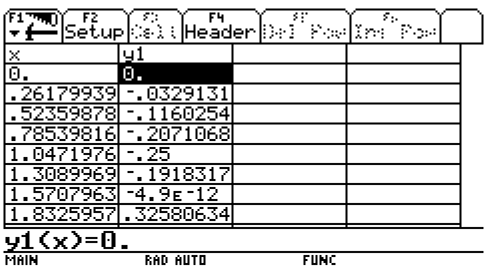


FIGURE 59

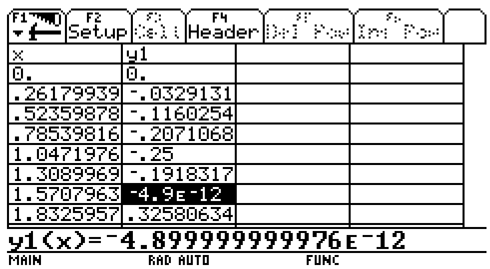


FIGURE 60

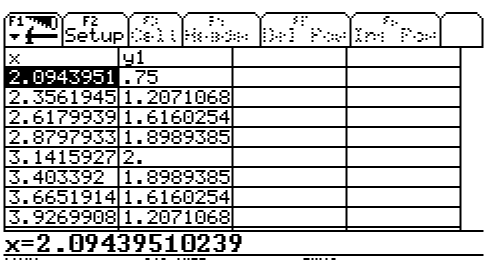


FIGURE 61

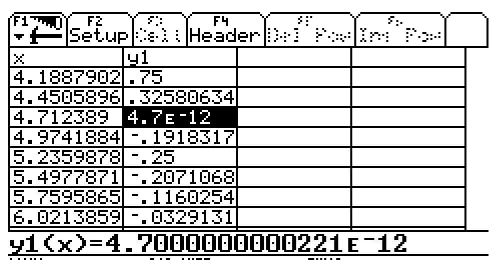


FIGURE 62

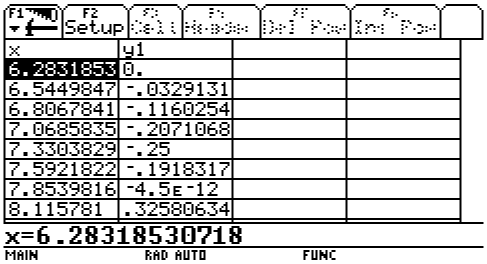


FIGURE 63

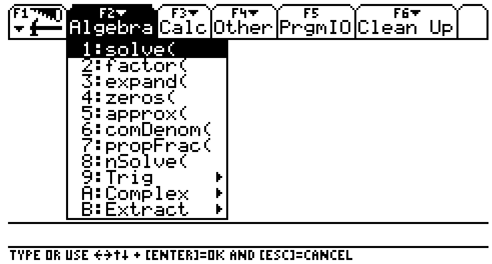


FIGURE 64

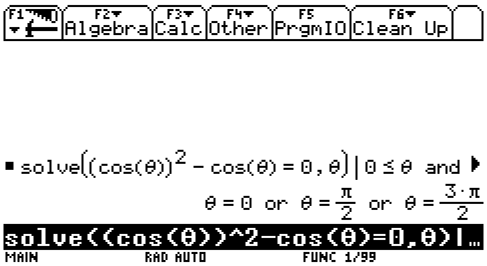


FIGURE 65

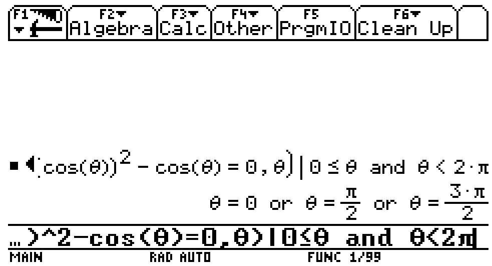


FIGURE 66

XI. Many students confuse number properties such as the commutative and associative properties. On the set of real numbers, can one find a binary operation that is: (a). commutative and associative; (b). not commutative and not associative; (c). associative but not commutative; (d). commutative but not associative. Hence the properties of commutativity and associativity are logically independent in the sense that neither one implies the other. Here are some examples:

(a). The binary operations of addition and multiplication on the set $\mathbb{Z}$ of integers is both commutative and associative. A more esoteric binary operation $*$ which is both commutative and associative is as follows: Let $a,b\in\mathbb{Z}$ such that $a*b=a+b+a\cdot b.$ To verify this, note that $b*a=b+a+b\cdot a.$ Since $a,b\in\mathbb{Z}$ which commute under ordinary addition and multiplication, the result is verified yielding $a*b=b*a.$ To show that $*$ is associative requires a bit more verification. Let $a,b,c\in\mathbb{Z}.$ Then

$$(a*b)*c=(a+b+a\cdot b)*c=(a+b+a\cdot b)+c+(a+b+a\cdot b)\cdot c=a+b+a\cdot b+c+a\cdot c+b\cdot c+a\cdot b\cdot c.$$

$$a*(b*c)=a+(b+c+b\cdot c)+a\cdot(b+c+b\cdot c)=a+b+c+b\cdot c+a\cdot b+a\cdot c+a\cdot b\cdot c.$$

Since addition is commutative and associative in $\mathbb{Z}$, upon comparing final entries in the two calculations, we can conclude that $(a*b)*c=a*(b*c).$

(b). The binary operations of subtraction and division in the set $\mathbb{Z}$ of ordinary integers are neither commutative nor associative and a counterexample suffices for each. Let $a=20,\ b=10\ and\ c=2.$ Then $a-b=20-10=10\neq-10=10-20=b-a.$ (Subtraction is not commutative). $(a-b)-c=(20-10)-2=10-2=8\neq12=20-8=20-(10-2)=a-(b-c).$ (Subtraction is not associative).

$a \div b = 20 \div 10 = 2 \neq \frac{1}{2} = 10 \div 20 = b \div a.$ (Division is not commutative).

$(a \div b) \div c = (20 \div 10) \div 2 = 2 \div 2 = 1 \neq 4 = 20 \div 5 = 20 \div (10 \div 2) = a \div (b \div c).$ (Division is not associative).

A more interesting example is that of exponentiation. We note that in the set of integers, this operation is neither commutative nor associative:

Let $a = 2,\ b = 3\ and\ c = 4.$ Then $a^b = 2^3 = 9 \neq 9 = 3^2 = b^a.$ (Exponentiation is not commutative).

Is there one non-trivial case where $a^b = b^a$? YES! Let $a = 4\ and\ b = 2.$ Then $a^b = 4^2 = 16 = 2^4 = b^a.$

$\left(a^b\right)^c = \left(2^3\right)^4 = 2^{3\cdot 4} = 2^{12} = 4096 \neq 2417851639229258349412352 = 2^{81} = 2^{\left(3^4\right)} = a^{\left(b^c\right)}.$
(Exponentiation is not associative).

(c). The operation $\circ$ which denotes function composition is associative, but not commutative.

To see that the composition of functions is not necessarily commutative, let $f\ and\ g$ be functions defined as follows: $f(x) = x^2\ and\ g(x) = x + 1.$ Then

$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1\ \forall x \in Domain\ of\ f = \mathbb{R}.$

$(f \circ g)(x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2 \cdot x + 1\ \forall x \in Domain\ of\ g = \mathbb{R}.$

The calculator can form the composition of two functions. See **FIGURES 67-71:**

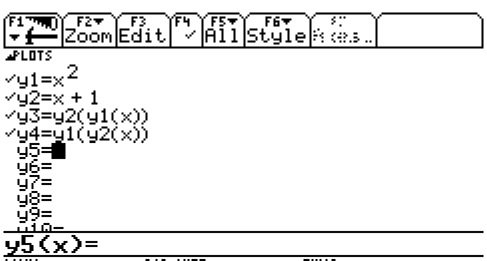


FIGURE 67

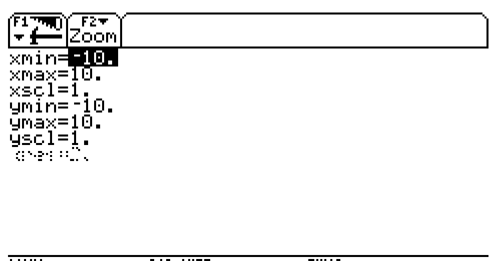
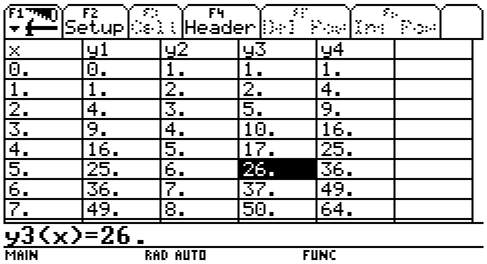
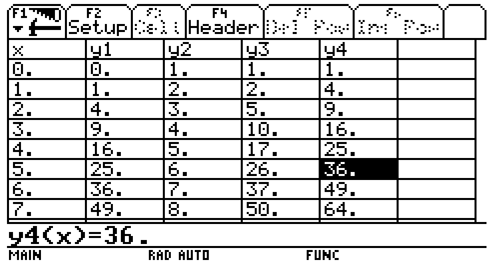


FIGURE 68



x	y1	y2	y3	y4	
0.	0.	1.	1.	1.	
1.	1.	2.	2.	4.	
2.	4.	3.	5.	9.	
3.	9.	4.	10.	16.	
4.	16.	5.	17.	25.	
5.	25.	6.	26.	36.	
6.	36.	7.	37.	49.	
7.	49.	8.	50.	64.	

FIGURE 69



x	y1	y2	y3	y4	
0.	0.	1.	1.	1.	
1.	1.	2.	2.	4.	
2.	4.	3.	5.	9.	
3.	9.	4.	10.	16.	
4.	16.	5.	17.	25.	
5.	25.	6.	26.	36.	
6.	36.	7.	37.	49.	
7.	49.	8.	50.	64.	

FIGURE 70

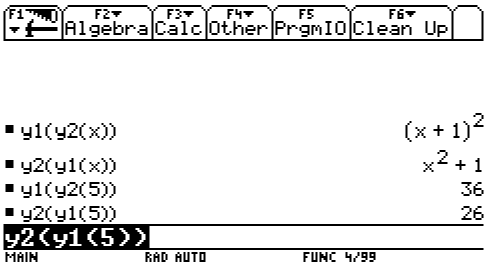


FIGURE 71

A numerical example now shows that these functions are different; for in most cases, different outputs will result from the same input. For example, let $x=5.$ Then
$(g\circ f)(5)=g\left(f(5)\right)=g\left(5^2\right)=g(25)=25+1=26\neq 36=6^2=f(6)=f(5+1)=f\left(g(5)\right)=(f\circ g)(5).$

Note that if f *and* g are a pair of inverse functions, then the functions commute. To cite an example, let $f(x)=2\cdot x+1$ *and* $g(x)=\dfrac{x-1}{2}$. Then

$$(f\circ g)(x)=f\left(g(x)\right)=f\left(\frac{x-1}{2}\right)=2\cdot\left(\frac{x-1}{2}+1\right)=x-1+1=x \text{ and}$$

$$(g\circ f)(x)=g\left(f(x)\right)=g(2\cdot x+1)=\left(\frac{2\cdot x+1-1}{2}\right)=\frac{2\cdot x}{2}=x.$$

(d). The following binary operation $*$ on $\mathbb{Z}$ is commutative, but not associative:

Let $*$ be defined as follows: If $a,b\in\mathbb{Z},$ *then* $a*b=a^2+b^2.$ Note that
$a*b=a^2+b^2=b^2+a^2=b*a.$ Since $a,b\in\mathbb{Z}\Rightarrow a^2,\ b^2\in\mathbb{Z}\Rightarrow a^2+b^2\in\mathbb{Z}\Rightarrow a^2+b^2=b^2+a^2.$
This establishes that $*$ is commutative. In contrast, $*$ is not associative. To see this, let $a=3,\ b=4$ *and* $c=5.$ Then

$$(a*b)*c=(3*4)*5=\left(3^2+4^2\right)*5=(9+16)*5=25*5=25^2+5^2=625+25=650\neq$$

$$1690=9+1681=3^2+41^2=3*41=3*(16+25)=3*\left(4^2+5^2\right)=3*(4*5)=a*(b*c).$$

XII. A student remarked that he found a prime number generating formula in the sense that all the output values are prime numbers: $p(n)=n^2-7\cdot n+53;\ n\in W=\{0,1,2,3,4,5,\ldots\}.$ By going far enough out in the sequence, prove him wrong! Here we must persevere in problem solving and use appropriate tools strategically.

In order to achieve our goal, we use a hand-held graphing calculator and consider the following inputs and outputs in **FIGURES 72-80:**

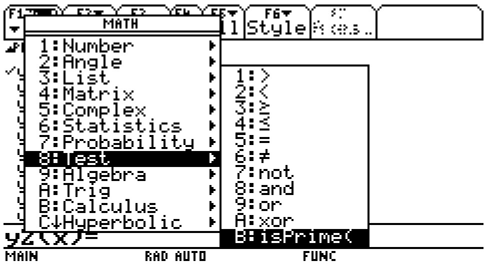


FIGURE 72

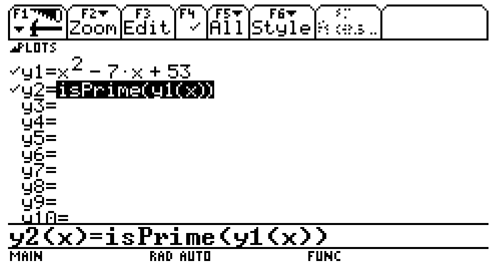


FIGURE 73

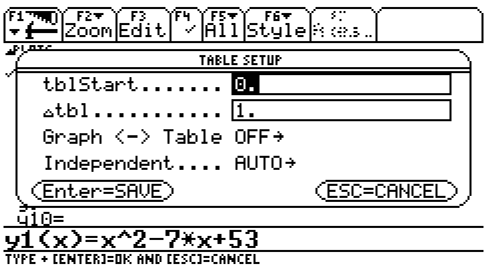
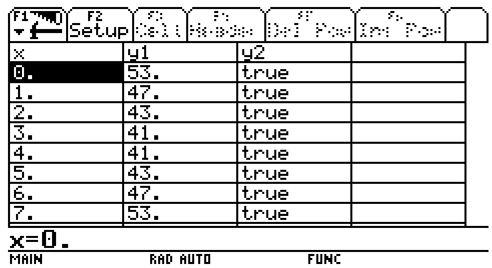
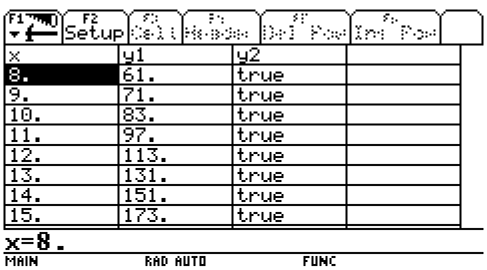


FIGURE 74



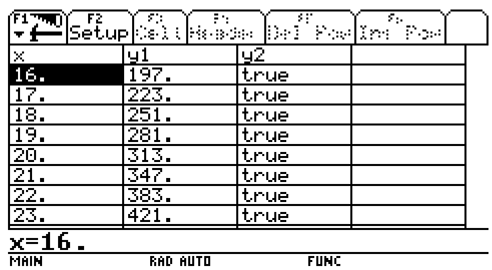
x	y1	y2
0.	53.	true
1.	47.	true
2.	43.	true
3.	41.	true
4.	41.	true
5.	43.	true
6.	47.	true
7.	53.	true

FIGURE 75



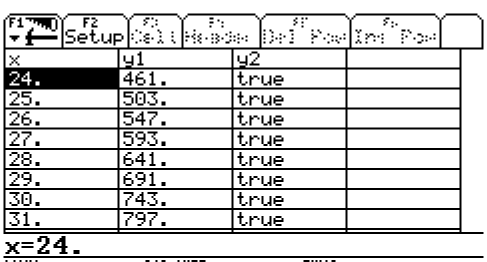
x	y1	y2
8.	61.	true
9.	71.	true
10.	83.	true
11.	97.	true
12.	113.	true
13.	131.	true
14.	151.	true
15.	173.	true

FIGURE 76



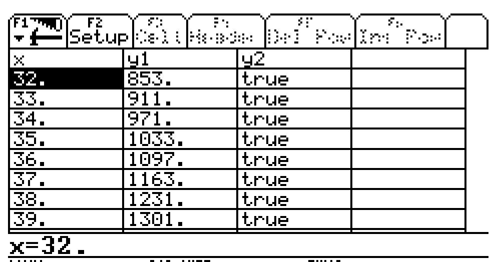
x	y1	y2
16.	197.	true
17.	223.	true
18.	251.	true
19.	281.	true
20.	313.	true
21.	347.	true
22.	383.	true
23.	421.	true

FIGURE 77



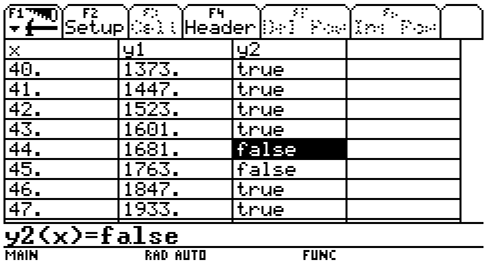
x	y1	y2
24.	461.	true
25.	503.	true
26.	547.	true
27.	593.	true
28.	641.	true
29.	691.	true
30.	743.	true
31.	797.	true

FIGURE 78



x	y1	y2
32.	853.	true
33.	911.	true
34.	971.	true
35.	1033.	true
36.	1097.	true
37.	1163.	true
38.	1231.	true
39.	1301.	true

FIGURE 79



x	y1	y2
40.	1373.	true
41.	1447.	true
42.	1523.	true
43.	1601.	true
44.	1681.	false
45.	1763.	false
46.	1847.	true
47.	1933.	true

y2(x)=false

FIGURE 80

THANK YOU FOR YOUR PARTICIPATION AT THIS WORKSHOP DURING THE THE 7TH ANNUAL KCM CONFERENCE ENGAGE 2015 IN LEXINGTON, KY!