

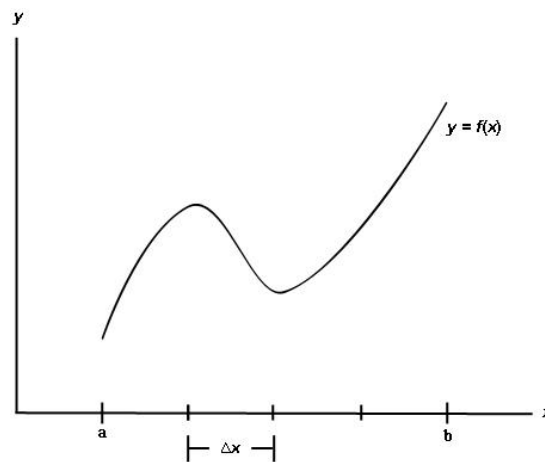
## CHAPTER 8:

# INTEGRALS

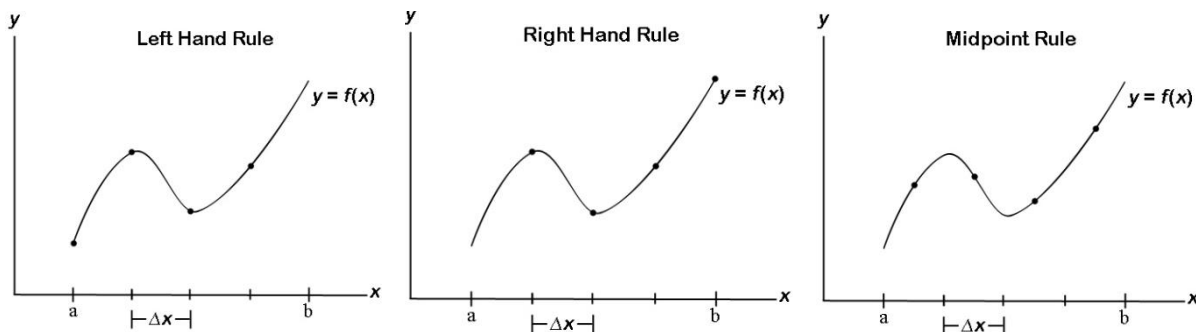
### 8.1 REVIEW: APPROXIMATING INTEGRALS WITH RIEMANN SUMS IN 2-D

In two dimensions we have previously used Riemann sums to approximate  $\int_a^b f(x)dx$  with the following steps:

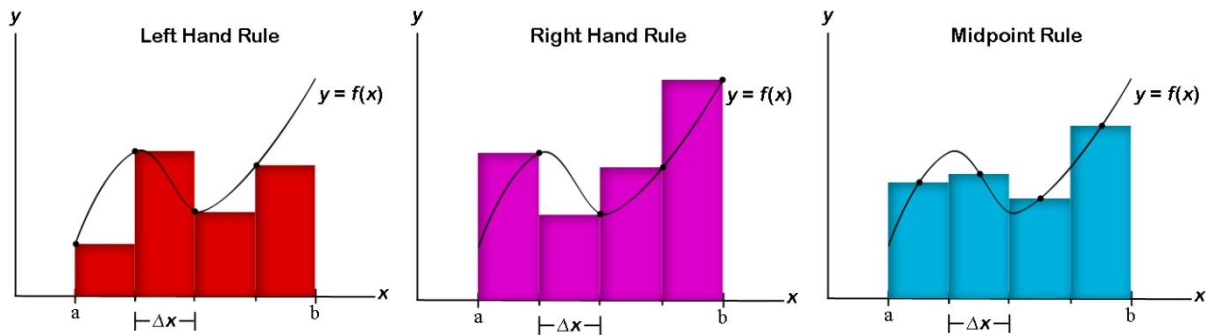
1. Divide the region  $a \leq x \leq b$  into intervals of width  $\Delta x$ .



2. For each interval, select the height of the function that will be used for that interval. There are various rules to choose the selected height for each interval, some of which are presented below.



3. For each interval, approximate the area under the curve over that particular interval with a rectangle with width  $\Delta x$  and height according to the indicated rule.



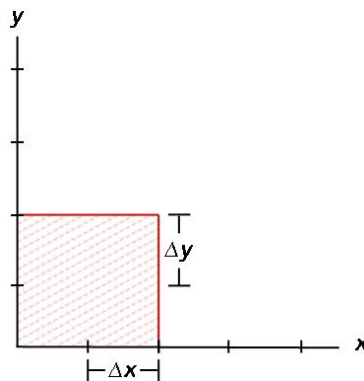
4. The approximate value of  $\int_a^b f(x)dx$  can be obtained by summing the areas of the rectangles according to the given rule.

## 8.2 APPROXIMATING INTEGRALS WITH RIEMANN SUMS IN 3D

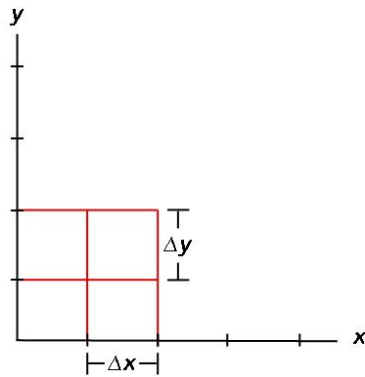
### VARIOUS RULES FOR RIEMANN SUMS IN 3D

To obtain the volume of a surface  $f(x, y)$  over an area  $A$ , a set of steps that are similar to those that were used in two dimensions can be used.

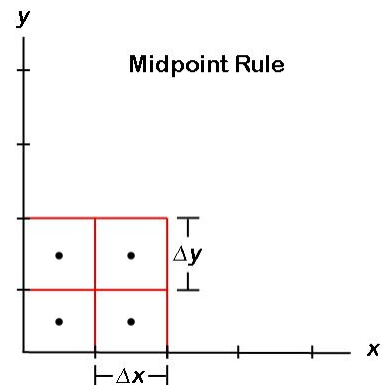
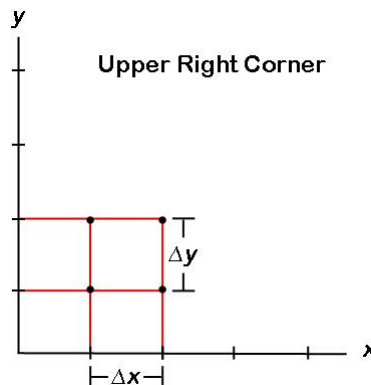
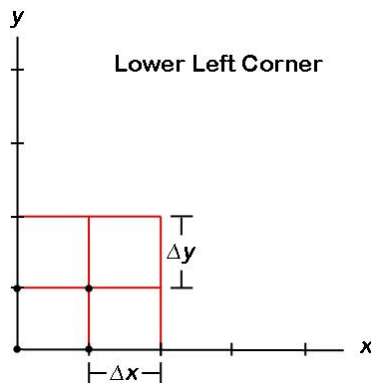
1. Identify Domain of  $f(x, y)$  to be evaluated.



2. Divide the domain into rectangles with length  $\Delta x$  and width  $\Delta y$ .



3. For each rectangle, find the height of the surface that will be used according to the indicated rule. There are various possible rules, a few of which are outlined below.

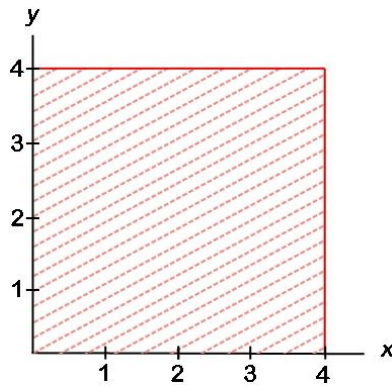


4. The approximate value of  $\int_A f(x,y) dA$  can be obtained by summing the volumes of the cubes as will be illustrated in the following examples.

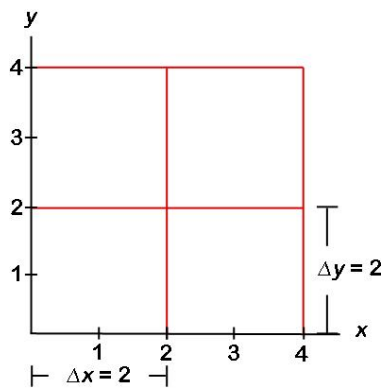
**Example Exercise 8.2.1:** Approximate  $\int_0^4 \int_0^4 (\frac{x^2}{2} + y^2 + 4) dy dx$  using rectangles of length  $\Delta x = 2$  and of width  $\Delta y = 2$  and using the height in the lower left hand corner of each division.

**Solution:**

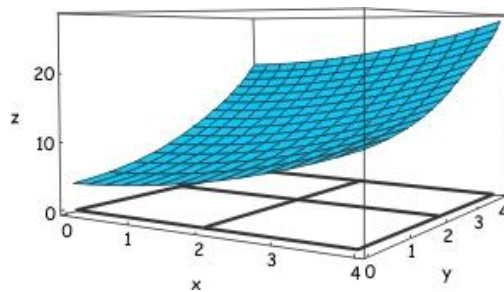
1. Identify Domain of  $f(x, y)$  to be evaluated.



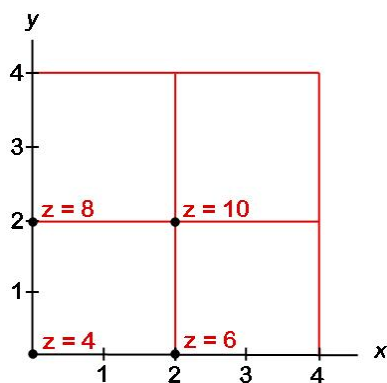
2. Divide the domain into rectangles with length  $\Delta x = 2$  and width  $\Delta y = 2$ .



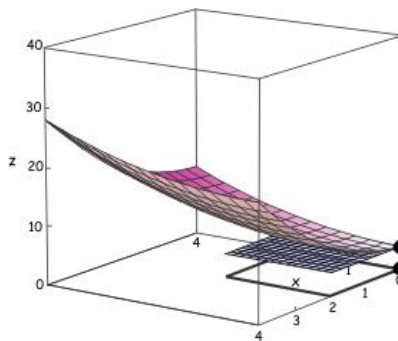
3. Place the surface over the region.



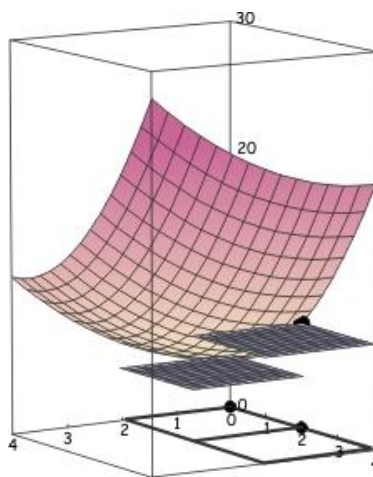
4. For each rectangle, find the height of the surface at the chosen point. In this case, find the value of  $f(x, y)$  in the lower left corner.



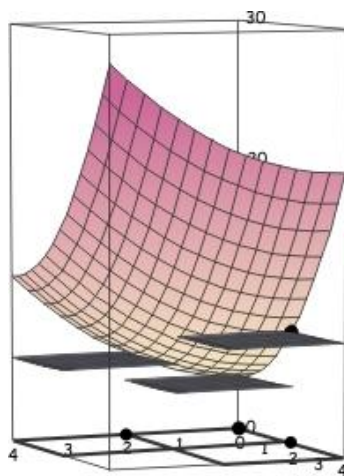
5. For the rectangle,  $0 \leq x \leq 2$ ,  $0 \leq y \leq 2$ , the height  $f(0, 0)$  in the left hand corner is  $z = 4$  so place the plane  $z = 4$  above this region.



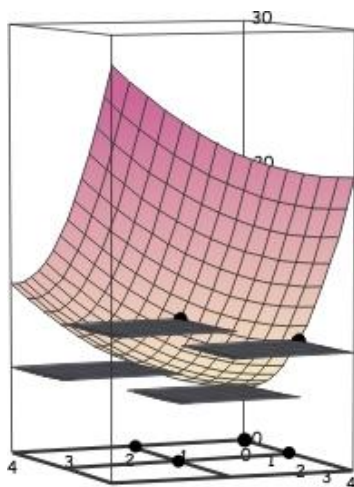
6. For the rectangle,  $0 \leq x \leq 2$ ,  $2 \leq y \leq 4$ , the height  $f(0, 2)$  in the left hand corner is  $z = 6$  so place the plane  $z = 6$  above this region.



7. For the rectangle,  $2 \leq x \leq 4$ ,  $0 \leq y \leq 2$ , the height  $f(0, 2)$  in the left hand corner is  $z = 8$  so place the plane  $z = 8$  above this region.



8. For the rectangle,  $2 \leq x \leq 4$ ,  $2 \leq y \leq 4$ , the height  $f(2, 2)$  in the left hand corner is  $z = 14$  so place the plane  $z = 14$  above the region.



9. For each rectangle, find the volume of the cuboid between the rectangle in the  $xy$  plane and the indicated plane above that region.

Rectangle	Length	Width	Height	Volume
1	2	2	4	$2 \times 2 \times 4 = 16$
2	2	2	6	$2 \times 2 \times 6 = 24$
3	2	2	8	$2 \times 2 \times 8 = 32$
4	2	2	10	$2 \times 2 \times 10 = 40$

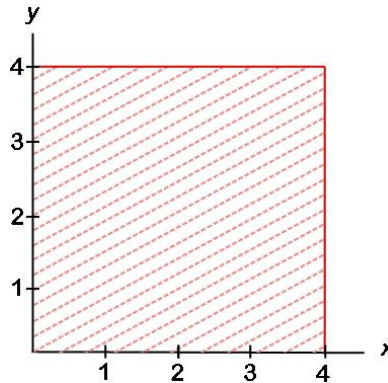
10. Approximate the volume beneath the surface and over the region as the sum of the volumes over the four rectangles

$$\text{Volume} = 16 + 24 + 32 + 40 = 112$$

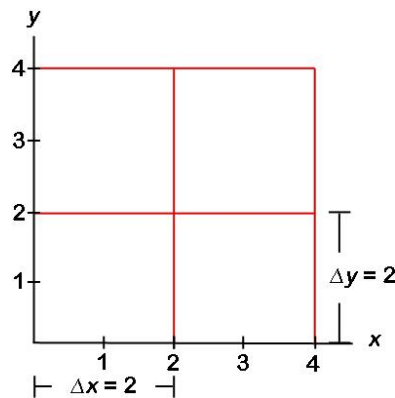
**Example Exercise 8.2.2:** Approximate  $\int_0^4 \int_0^4 (\frac{x^2}{2} + y^2 + 4) dy dx$  using rectangles of length  $\Delta x = 2$  and of width  $\Delta y = 2$  and using the height in the middle of each division.

**Solution:**

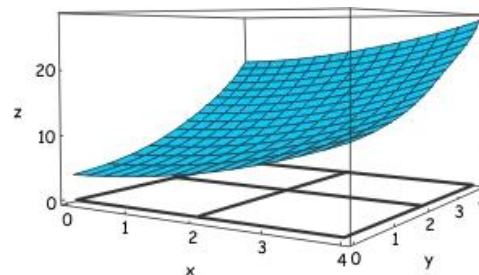
1. Identify Domain of  $f(x, y)$  to be evaluated.



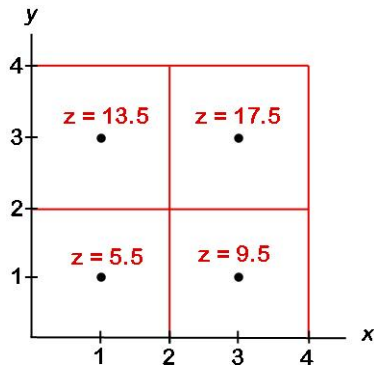
2. Divide the domain into rectangles with length  $\Delta x = 2$  and width  $\Delta y = 2$ .



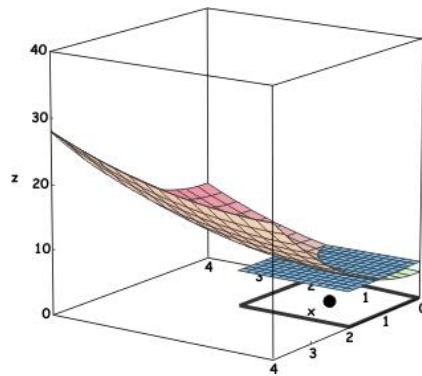
3. Place the surface over the region.



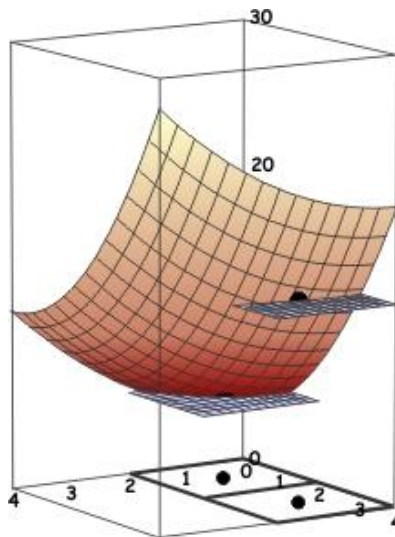
4. For each rectangle, find the height of the surface at the chosen point. In this case, find the value of  $f(x, y)$  in the middle of the rectangle.



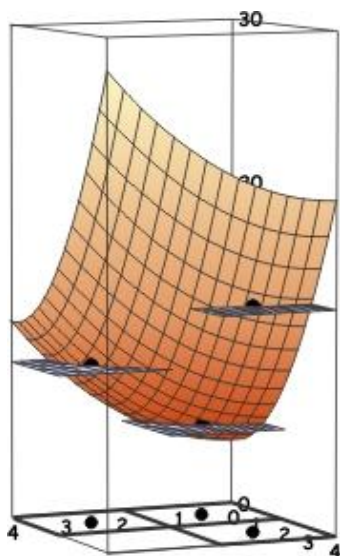
5. For the rectangle,  $0 \leq x \leq 2$ ,  $0 \leq y \leq 2$ , the height  $f(1, 1) = 5.5$  so place the plane  $z = 5.5$  above this region.



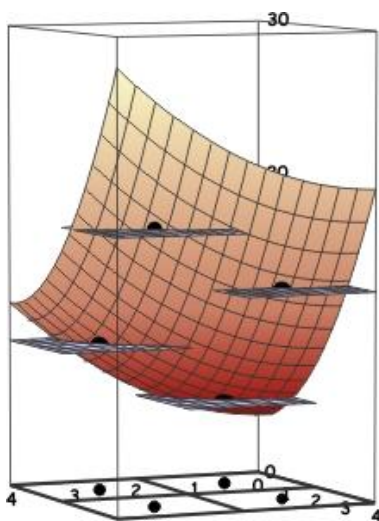
6. For the rectangle  $0 \leq x \leq 2$ ,  $2 \leq y \leq 4$ , the height in the middle of the rectangle is  $f(1, 3) = 13.5$  so place the plane  $z = 13.5$  above this region.



7. For the rectangle  $2 \leq x \leq 4$ ,  $0 \leq y \leq 2$ , the height in the middle of the rectangle is  $f(3, 1) = 9.5$  so place the plane  $z = 9.5$  above this region.



8. For the rectangle  $2 \leq x \leq 4$ ,  $2 \leq y \leq 4$ , the height in the middle of the rectangle is  $f(3, 3) = 17.5$  so place the plane  $z = 17.5$  above this region.



9. For each rectangle, find the volume of the cuboid between the rectangle in the  $xy$  plane and the indicated plane above that region.

Rectangle	Length	Width	Height	Volume
1	2	2	5.5	$2 \times 2 \times 5.5 = 22$
2	2	2	9.5	$2 \times 2 \times 9.5 = 38$
3	2	2	13.5	$2 \times 2 \times 13.5 = 54$
4	2	2	17.5	$2 \times 2 \times 17.5 = 70$

10. Approximate the volume beneath the surface and over the region as the sum of the volumes over the four rectangles.

$$\text{Volume} = 22 + 38 + 54 + 70 = 184$$

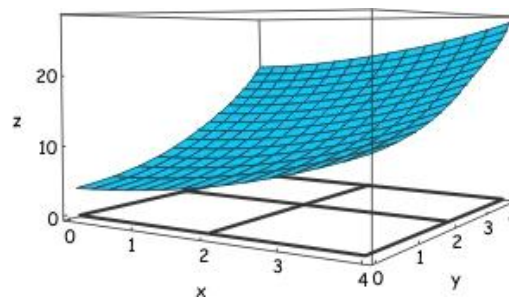
### 8.3 USING RIEMANN SUMS AND THE FUNDAMENTAL THEOREM TO FIND PRECISE VOLUMES

To understand why double integrals can be used to represent the volume below a surface, we can use Riemann sums and the fundamental theorem of the Integral Calculus as shown in the following example:

**Example Exercise 8.3.1:** The formula to represent a function  $f$  with inputs  $x$  and  $y$  and output  $z = f(x, y) = 0.5x^2 + y^2 + 4$ . We wish to obtain the volume below  $f$  and over the region  $0 \leq x \leq 4, 0 \leq y \leq 4$  in the  $xy$  plane. The units of  $x, y$  and  $z$  are all inches.

**Solution:**

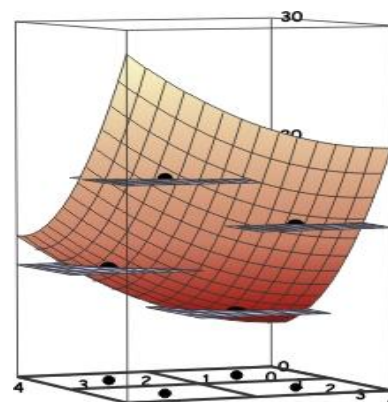
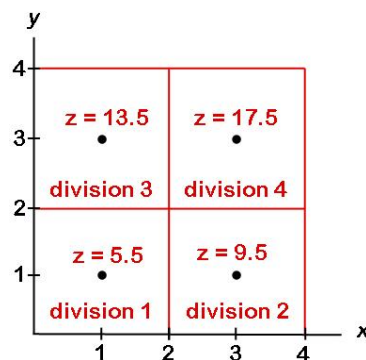
A. Place the surface and identify the volume we're seeking in your 3D kit.



B. If there are two divisions in  $x$  and two divisions in  $y$  and the volume is to be approximated using the middle value for  $x$  and  $y$  in each division, find  $x_1, x_2, y_1, y_2, \Delta x$ , and  $\Delta y$

$$x_1 = 1, x_2 = 3, y_1 = 1, y_2 = 3, \Delta x = 2 \text{ and } \Delta y = 2$$

C. For each of the four divisions on your kit, indicate the point and associated height we will use to approximate the volume for that division.



D. Use the values obtained in part C to fill in the following table with numerical values.

Division	Length	Width	Height	Volume
1	2	2	5.5	2 x 2 x 5.5
2	2	2	9.5	2 x 2 x 9.5
3	2	2	13.5	2 x 2 x 13.5
4	2	2	17.5	2 x 2 x 17.5

E. Fill in the same table below using  $x_1, x_2, y_1, \Delta x$ , and  $\Delta y$  instead of numerical values. (Note, the divisions should not change between the two tables.)

Division	Length	Width	Height	Volume
1	$\Delta y$	$\Delta x$	$0.5x_1^2 + y_1^2 + 4$	$(0.5x_1^2 + y_1^2 + 4)\Delta x\Delta y$
2	$\Delta y$	$\Delta x$	$0.5x_2^2 + y_1^2 + 4$	$(0.5x_2^2 + y_1^2 + 4)\Delta x\Delta y$
3	$\Delta y$	$\Delta x$	$0.5x_1^2 + y_2^2 + 4$	$(0.5x_1^2 + y_2^2 + 4)\Delta x\Delta y$
4	$\Delta y$	$\Delta x$	$0.5x_2^2 + y_2^2 + 4$	$(0.5x_2^2 + y_2^2 + 4)\Delta x\Delta y$

F. Express the approximate volume numerically.

$$\text{Volume} \approx 2 \times 2 \times 5.5 + 2 \times 2 \times 9.5 + 2 \times 2 \times 13.5 + 2 \times 2 \times 17.5 = 184$$

G. Express the same volume obtained in part F using  $x_1, x_2, y_1, y_2, \Delta x$ , and  $\Delta y$

$$(0.5x_1^2 + y_1^2 + 4)\Delta x\Delta y + (0.5x_2^2 + y_1^2 + 4)\Delta x\Delta y + (0.5x_1^2 + y_2^2 + 4)\Delta x\Delta y + (0.5x_2^2 + y_2^2 + 4)\Delta x\Delta y$$

H. Express the volume obtained in part G in the form  $\sum \sum (...)\Delta y\Delta x$

$$(0.5x_1^2 + y_1^2 + 4)\Delta x\Delta y + (0.5x_2^2 + y_1^2 + 4)\Delta x\Delta y + (0.5x_1^2 + y_2^2 + 4)\Delta x\Delta y + (0.5x_2^2 + y_2^2 + 4)\Delta x\Delta y$$

$$= \sum_{j=1}^2 (0.5x_1^2 + y_j^2 + 4) \Delta y\Delta x + \sum_{j=1}^2 (0.5x_2^2 + y_j^2 + 4) \Delta y\Delta x$$

$$= \sum_{i=1}^2 \sum_{j=1}^2 (0.5x_i^2 + y_j^2 + 4) \Delta y\Delta x$$

I. Use the fundamental theorem and limits to convert the approximation in part H to a precise value.

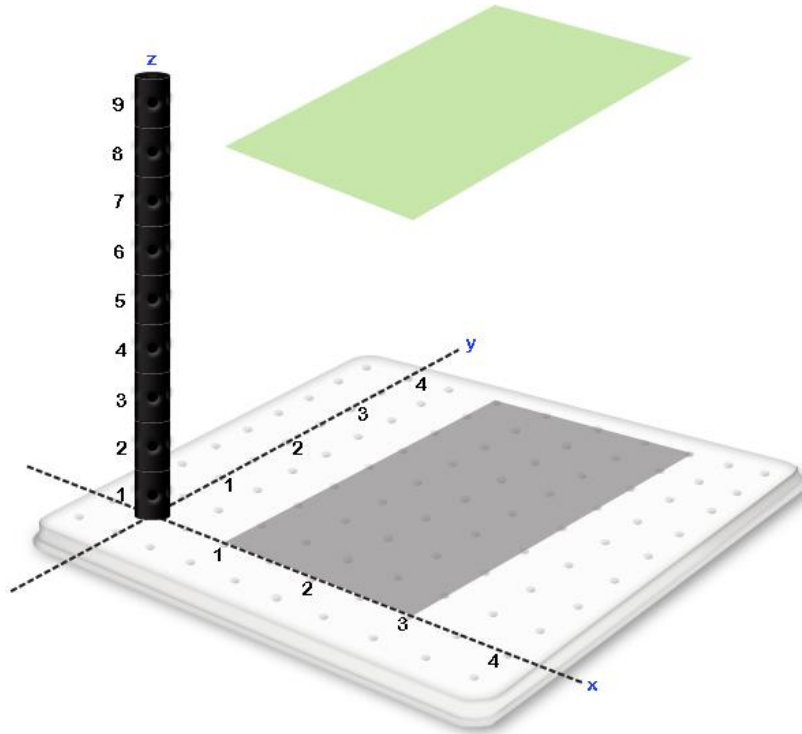
$$\text{Volume} = \lim_{\substack{\Delta x \rightarrow 0, \\ \Delta y \rightarrow 0}} \sum_{i=1}^n \sum_{j=1}^m (0.5x_i^2 + y_j^2 + 4) \Delta y\Delta x = \int_0^4 \int_0^4 (0.5x^2 + y^2 + 4) dydx$$

## 8.4 DEFINITION OF THE INTEGRAL

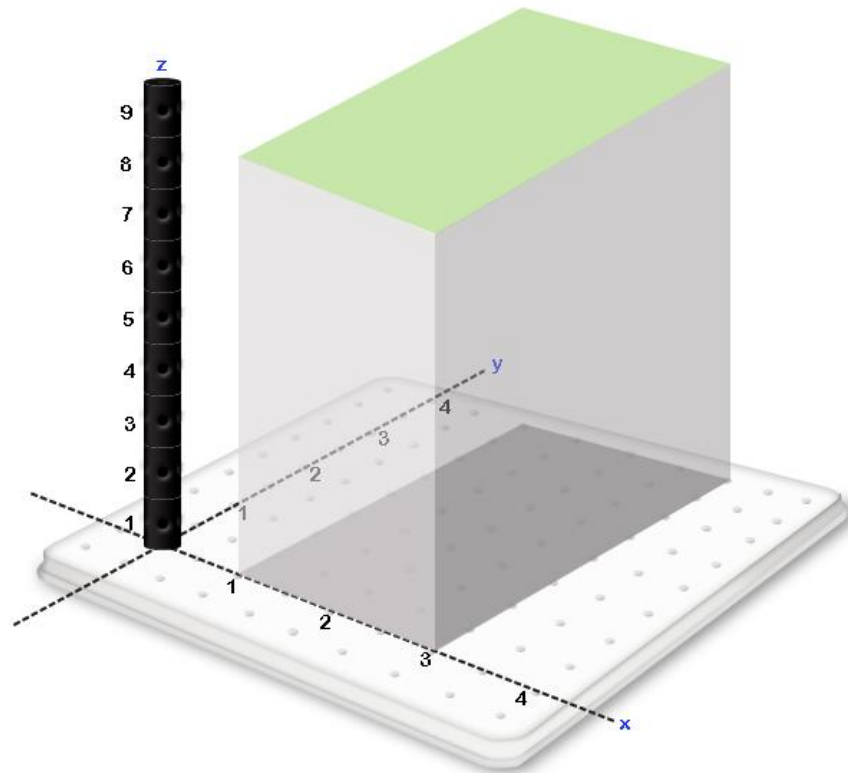
Given a surface  $z = f(x, y)$  and a region in the  $xy$  plane  $A$ , the volume below the surface of  $z = f(x, y)$  and above the region  $A$  in the  $xy$  plane is defined as  $\int_A f(x, y) dA$ .

### *Example*

If  $A$  is the rectangle  $1 \leq x \leq 3$ ,  $0 \leq y \leq 4$  and  $f(x, y) = 8$ .

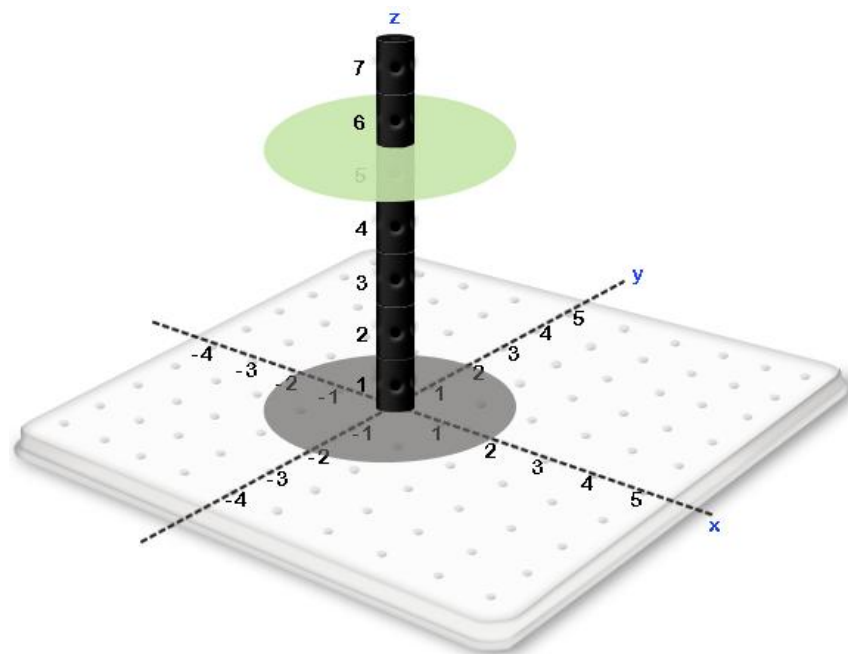


As can be seen, the volume associated with  $\int_A f(x, y) dA$  is a cube with length 2, width 4, and height 8 will give us a volume of 64.

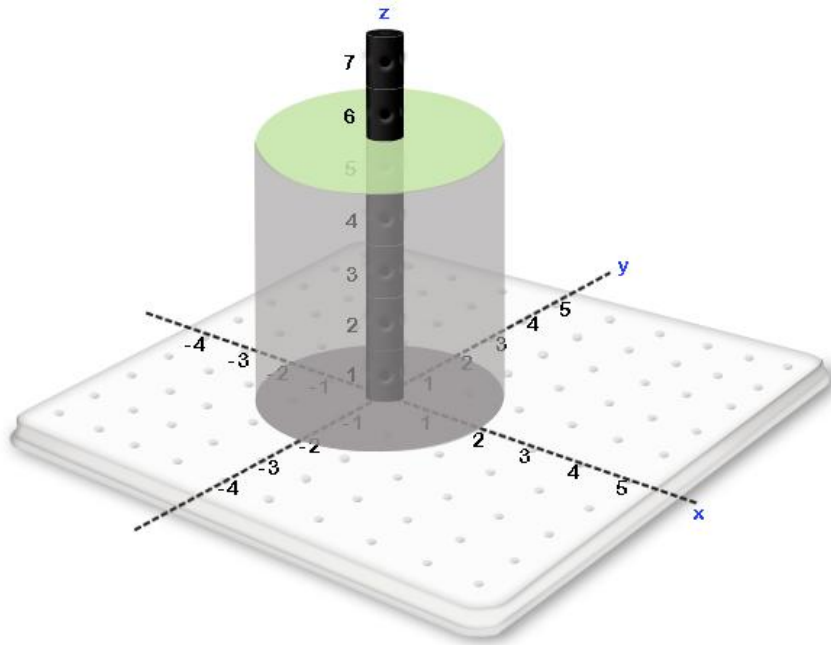


*Example*

If  $A$  is the region  $x^2 + y^2 \leq 4$  and  $f(x, y) = 5$ .



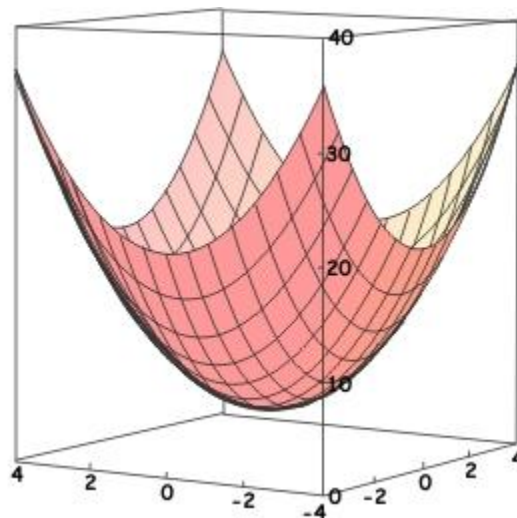
As can be seen, the volume associated with  $\int_A f(x, y) dA$  is a cylinder with base radius 2 and height 5 which will give us a volume of  $20\pi r^2$



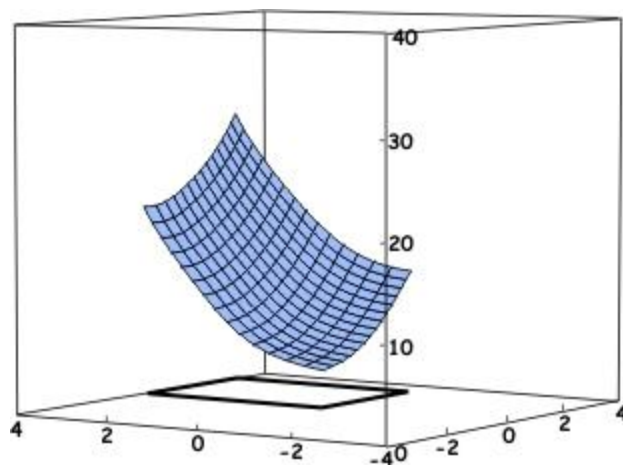
### Example

If A is the rectangle  $0 \leq x \leq 3$ ,  $0 \leq y \leq 4$  and  $f(x, y) = x^2 + y^2$ .

The surface in general is the following:



The surface over the indicated range is the following:



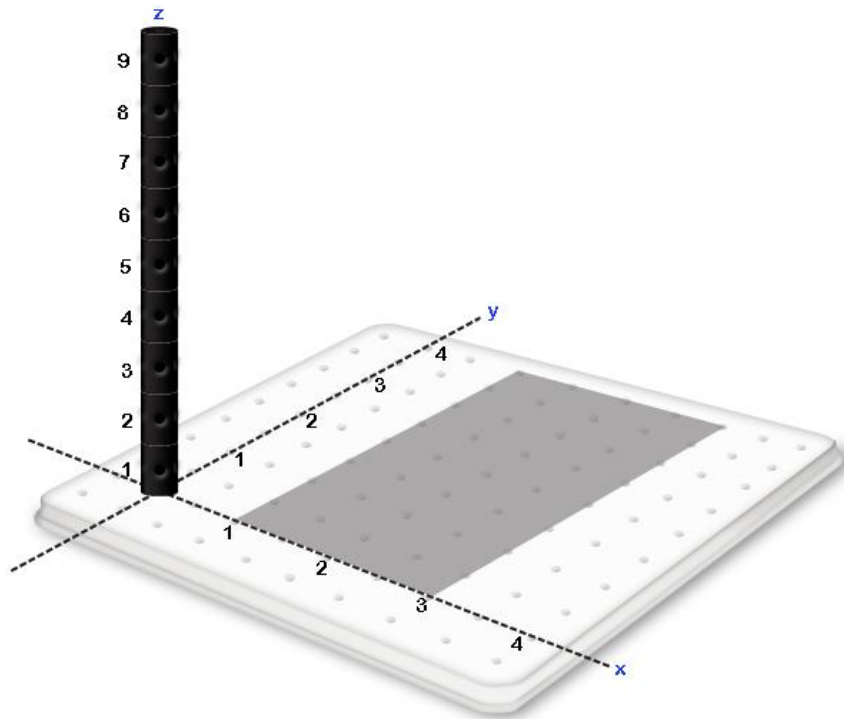
As can be seen, the volume associated with  $\int_A f(x,y)dA$  is not a standard volume and correspondingly cannot be obtained immediately with known formulas.

## 8.5 IDENTIFYING VOLUMES ASSOCIATED WITH DOUBLE INTEGRALS

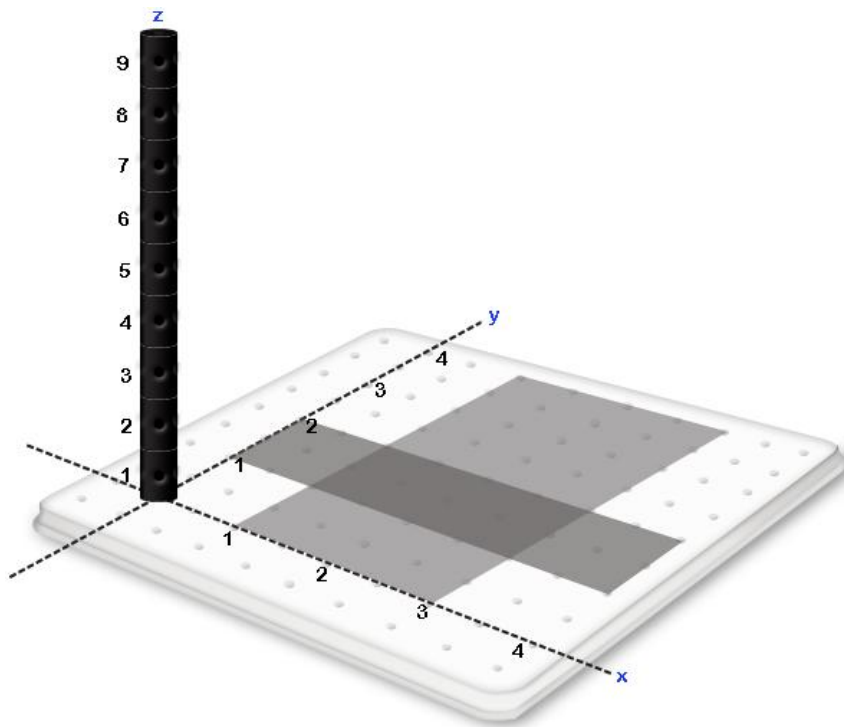
Traditionally, instead of  $\int_A f(x,y)dA$ , a double integral is used to represent volumes. The following examples, illustrate how to associate a double integral with its corresponding volume.

**Example Exercise 8.5.1:** Given the integral  $\int_1^3 \int_1^2 (2x + y + 1) dy dx$ , identify the volume associated with the integral.

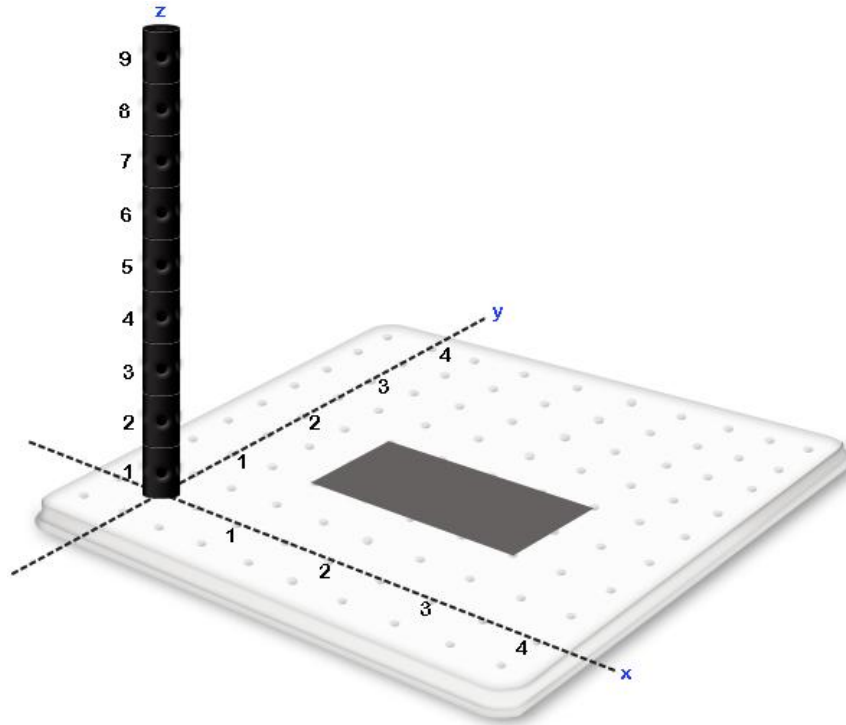
**Solution:** Working from the outside inward, the outside integral is  $\int_1^3 \dots dx$  indicating that our volume will contain values of  $f$  between  $x = 1$  and  $x = 3$ .



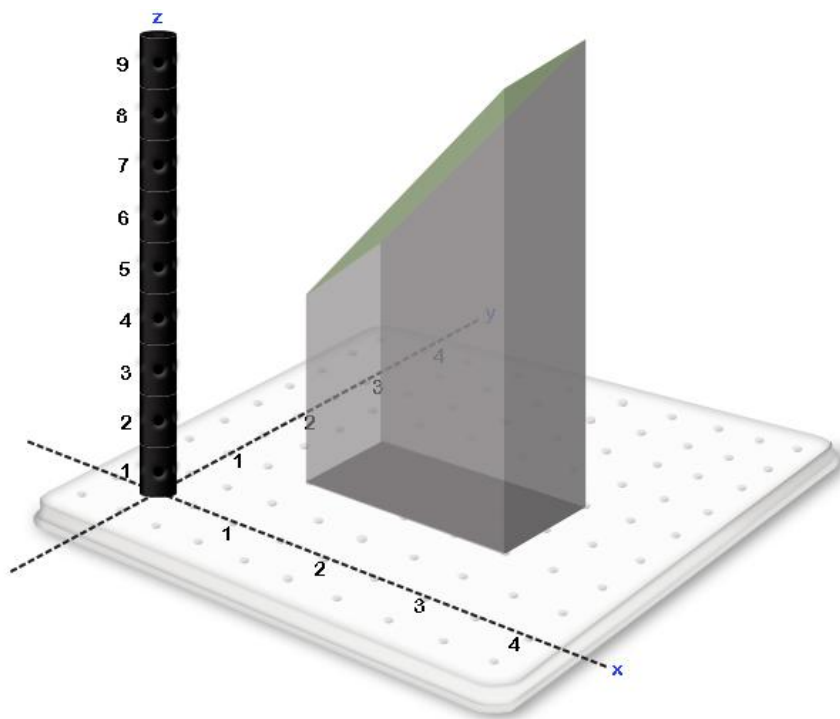
The inside integral is  $\int_1^2 \dots dy$  indicating that for every value of  $x$  between  $x = 1$  and  $x = 3$ , we will accept values of  $y$  that reside between  $y = 1$  and  $y = 2$ .



In the previous section we indicated that given a surface  $z = f(x, y)$  and a region in the  $xy$  plane  $A$ , the volume below the surface of  $z = f(x, y)$  and above the region  $A$  in the  $xy$  plane is defined as  $\int_A f(x, y) dA$ . The combination of the two integrals  $\int_1^3 \int_1^2 \dots dy dx$  defined thus far give a rectangular region in the  $xy$  plane which corresponds to  $\int_A \dots dA$  defined in the previous section.

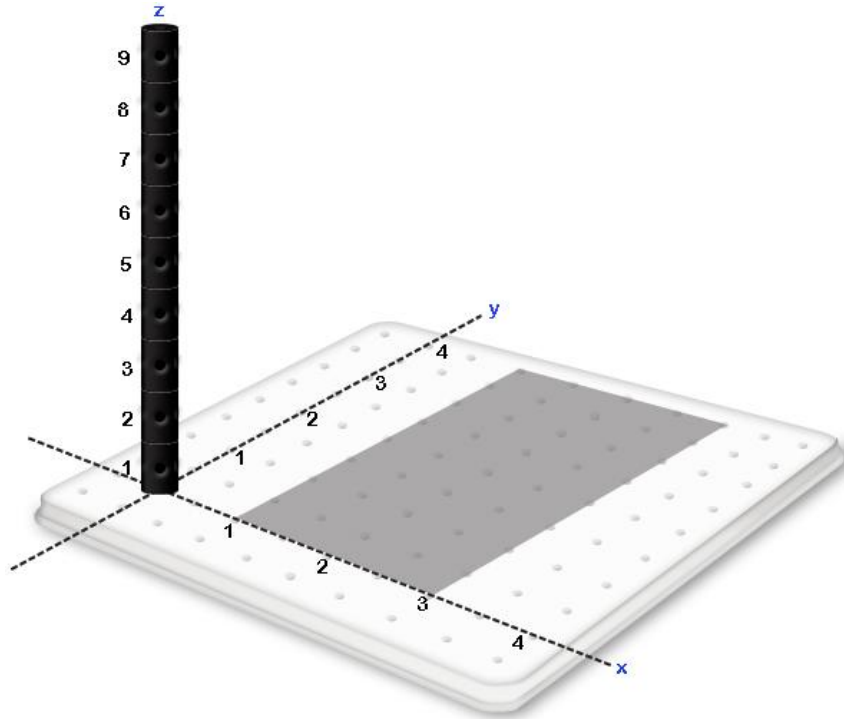


The volume associated with  $\int_1^3 \int_1^2 (2x + y + 1) dy dx$  is given by the volume above this region in the  $xy$  plane and below  $z = 2x + y + 1$ .

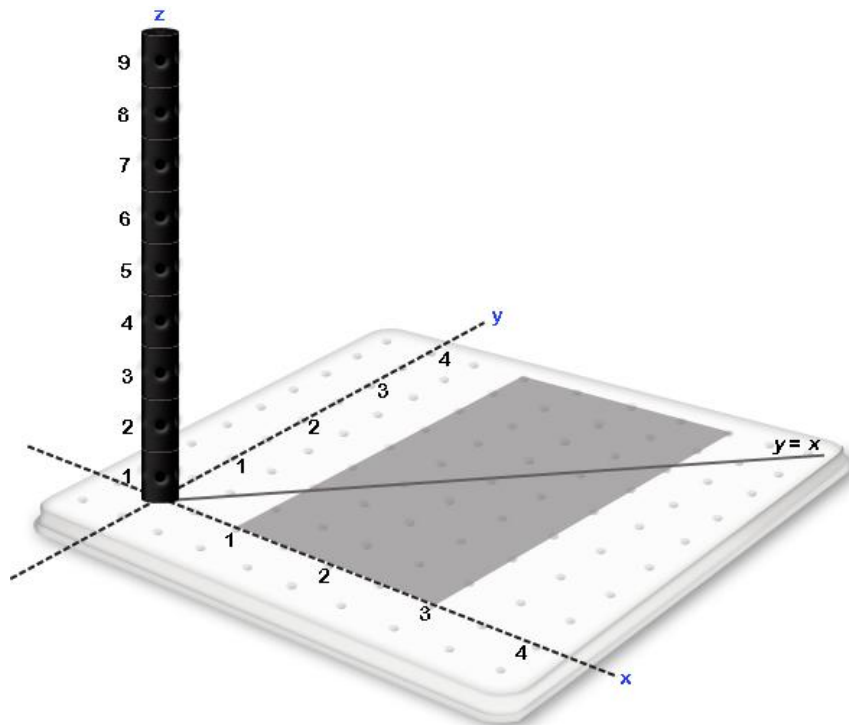


**Example Exercise 8.5.2:** Given the integral  $\int_1^3 \int_0^x (x + 3y + 1) dy dx$ , identify the volume associated with the integral.

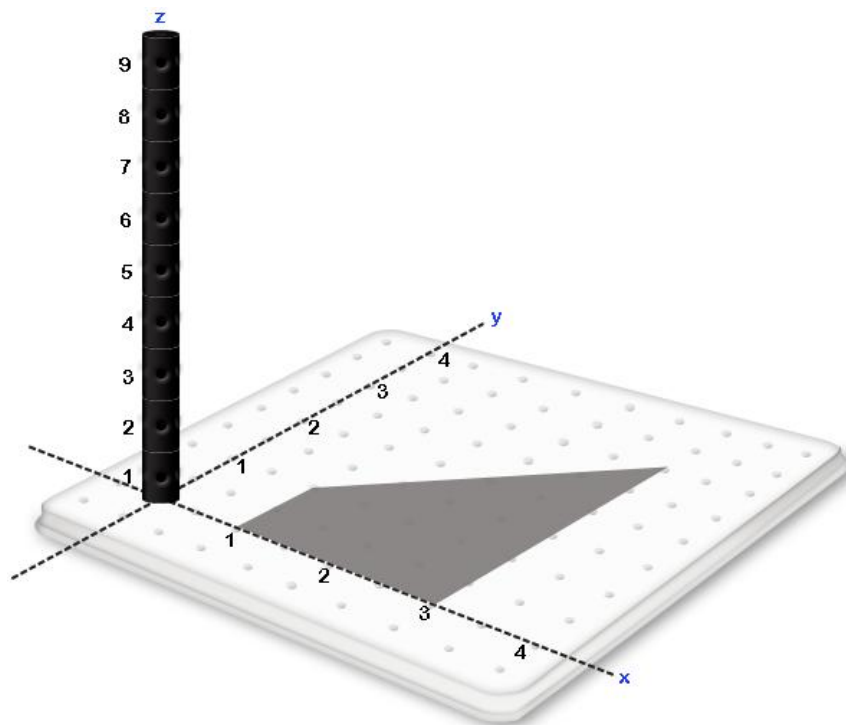
**Solution:** Working from the outside inward, the outside integral is  $\int_1^3 \dots dx$  indicating that our volume will contain values of  $x$  between  $x = 1$  and  $x = 3$ .



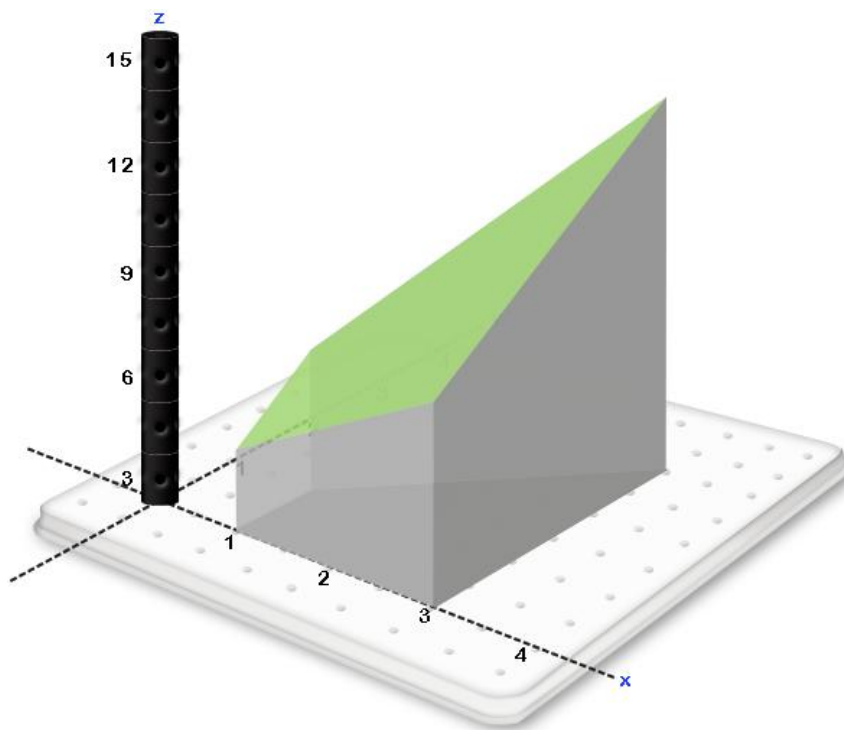
The second datum from the integral is  $\int_0^x \dots dy$  indicating that for every value of  $x$  between  $x = 1$  and  $x = 3$ , we will accept values of  $y$  that reside between  $y = 0$  and  $y = x$ .



In the previous section we indicated that given a surface  $z = f(x, y)$  and a region in the  $xy$  plane  $A$ , the volume below the surface of  $z = f(x, y)$  and above the region  $A$  in the  $xy$  plane is defined as  $\int_A f(x, y) dA$ . The combination of the two integrals  $\int_1^3 \int_0^x \dots dy dx$  defined thus far give a trapezoidal region in the  $xy$  plane which corresponds to  $\int_A \dots dA$  defined in the previous section.

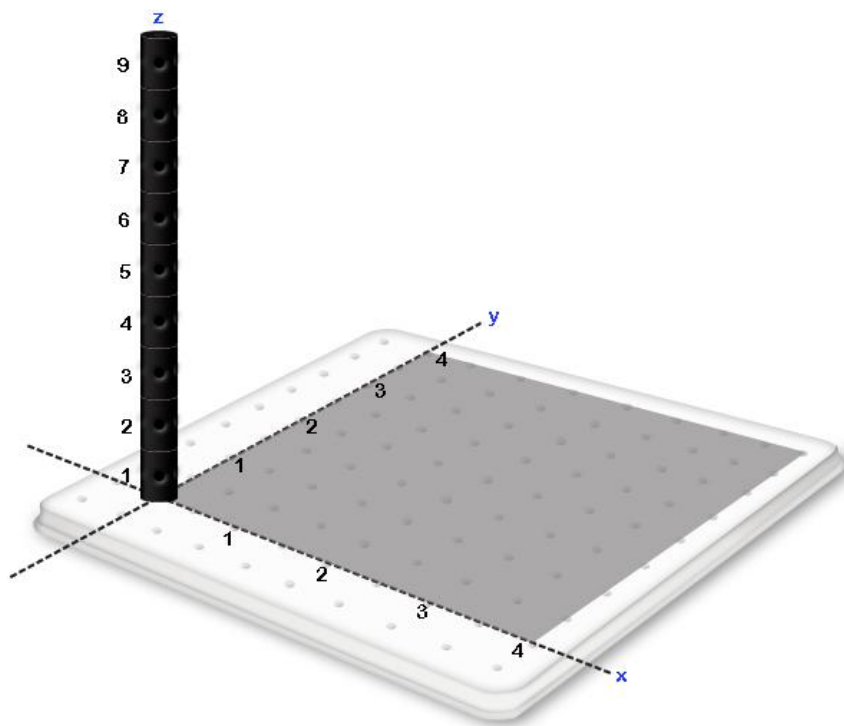


The volume associated with  $\int_1^3 \int_0^x (x + 3y + 1) dy dx$  is given by the volume above this region in the  $xy$  plane and below the plane  $z = x + 3y + 2$ .

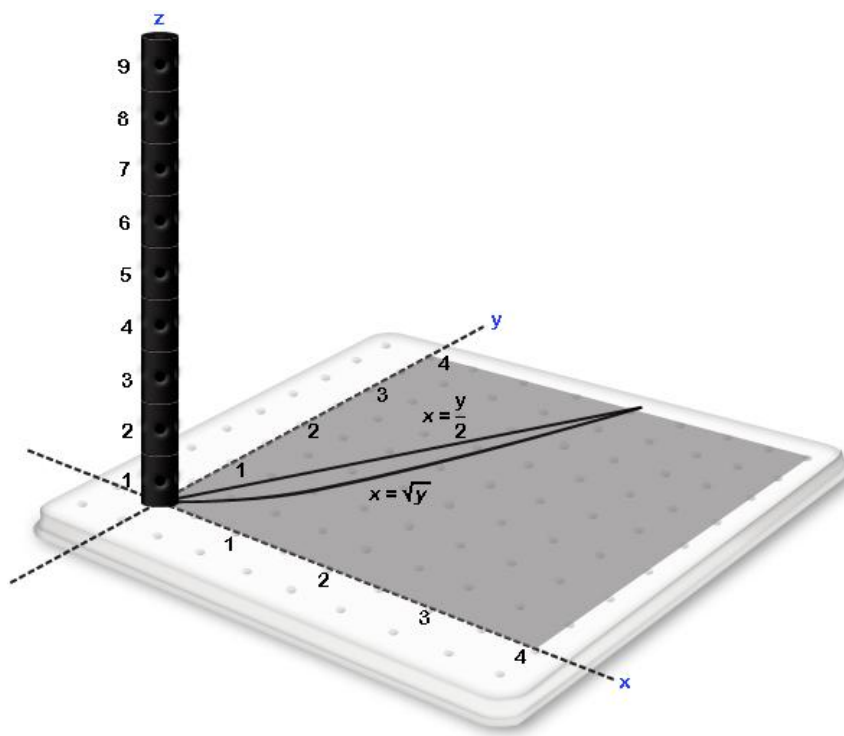


**Example Exercise 8.5.3:** Given the integral  $\int_0^4 \int_y^{\sqrt{y}} (x + y + 2) dx dy$ , identify the volume associated with the integral.

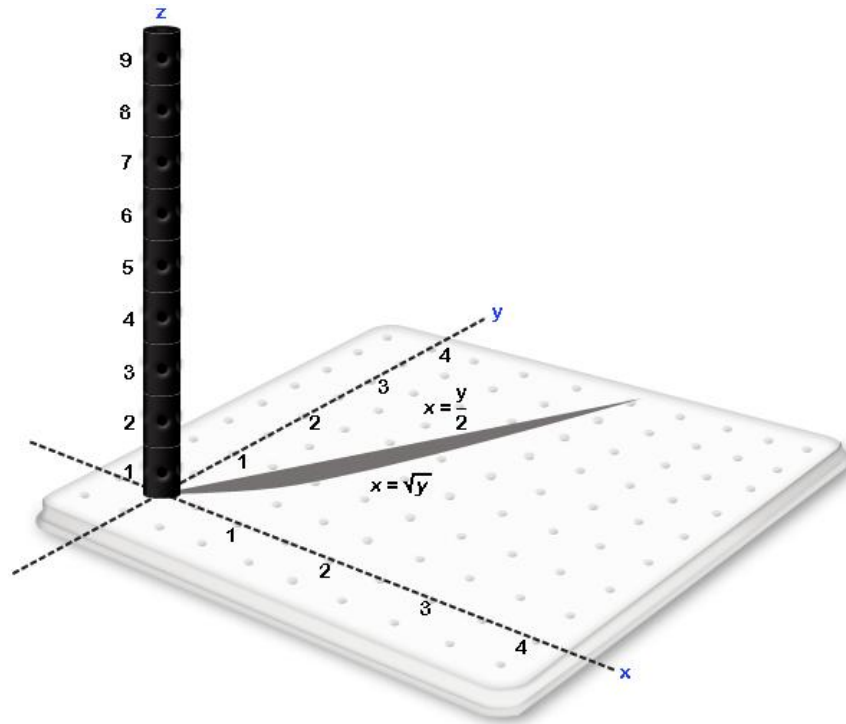
**Solution:** Working from the outside inward, the first datum from the integral is  $\int_0^4 \dots dy$  indicating that our volume will contain values of  $y$  between  $y = 0$  and  $y = 4$ .



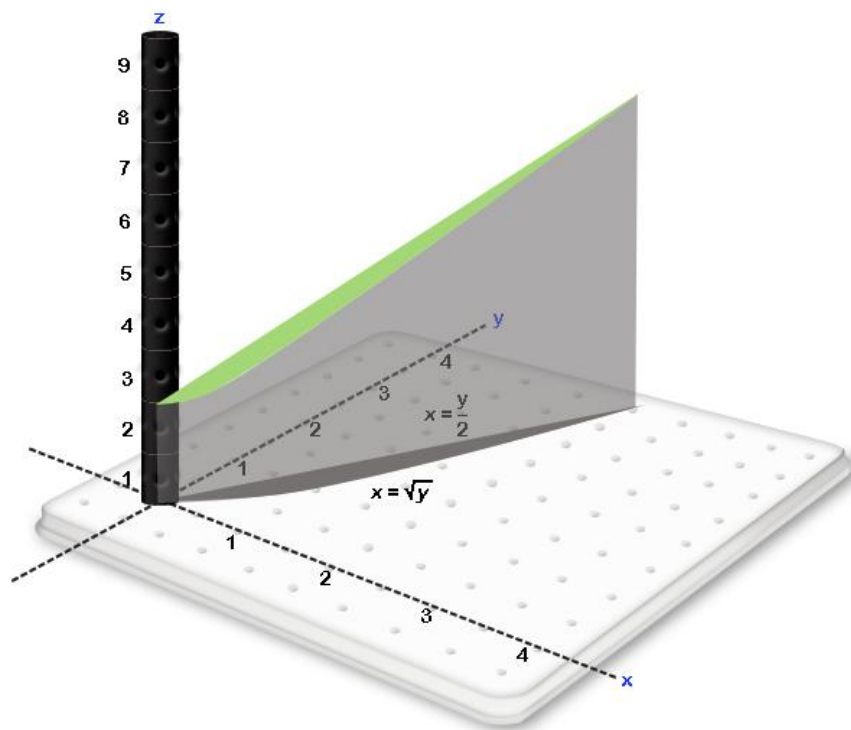
The second datum from the integral is  $\int_{\frac{y}{2}}^{\sqrt{y}} \dots dA$  indicating that for every value of  $y$  between  $y = 1$  and  $y = 4$ , we will accept values of  $x$  that reside between  $x = \frac{y}{2}$  and  $x = \sqrt{y}$ .



In the previous section we indicated that gives a surface  $z = f(x, y)$  and a region in the  $xy$  plane  $A$ , the volume below the surface of  $z = f(x, y)$  and above the region  $A$  in the  $xy$  plane is defined as  $\int_A f(x, y) dA$ . The combination of the two integrals  $\int_0^4 \int_y^{\sqrt{y}} \dots dx dy$  defined thus far give a region in the  $xy$  plane which corresponds to  $\int_A \dots dA$  defined in the previous section



The volume associated with  $\int_0^4 \int_y^{\sqrt{y}} (x + y + 2) dx dy$  is given by the volume above this region in the  $xy$  plane and below the plane  $z = x + y + 2$ .



### EXERCISE PROBLEMS:

- 1) A. Place the volume between  $x = 0$ ;  $x = 2$ ,  $y = 0$ ;  $y = 2x$ ;  $z = 0$ ; and  $z = x^2 + y^2$  on your 3D kit.  
 B. Express the volume with an integral of the form  $\iint \dots dydx$ .  
 C. Express the volume with an integral of the form  $\iint \dots dx dy$ .  
 D. Evaluate the two integrals to verify that they are the same.
  
- 2) A. Place the volume between  $x = 0$ ;  $x = 2$ ,  $y = x^2$ ;  $y = 4$ ;  $z = 0$ ; and  $z = x^2 + y^2$  on your 3D kit.  
 B. Express the volume with an integral of the form  $\iint \dots dydx$ .  
 C. Express the volume with an integral of the form  $\iint \dots dx dy$ .  
 D. Evaluate the two integrals to verify that they are the same.
  
- 3)  $z = f(x, y)$  is represented by the following table:

	$x = 0$	$x = 2$	$x = 4$
$y = 0$	1	2	3
$y = 2$	3	3	3
$y = 4$	3	4	5

- A. Use the small magnets to construct the cuboids we will use to approximate  $\int_0^4 \int_0^4 f(x, y) dy dx$  using Riemann Sums with the Lower left corner rule with  $\Delta x = 2$  and  $\Delta y = 2$ . Then find the approximation for the volume.
- B. Use the small magnets to construct the cuboids we will use to approximate  $\int_0^4 \int_0^4 f(x, y) dy dx$  using Riemann Sums with the Upper Right corner rule with  $\Delta x = 2$  and  $\Delta y = 2$ . Then find the approximation for the volume.
- C. Express the volume obtained in B in the form  $\sum \sum (...) \Delta y \Delta x$ .

4) The formula to represent a function  $f$  with inputs  $x$  and  $y$  and output  $z$  is  $z = f(x, y) = 8 - x^2 - y^2$ .

- A. Use the small magnets to construct the cuboids we will use to approximate  $\int_0^4 \int_0^4 f(x, y) dy dx$  using Riemann Sums with the Midpoint rule with  $\Delta x = 2$  and  $\Delta y = 2$ . Then find the approximation for the volume.
- B. Express the volume in A in the form  $\sum \sum (...) \Delta y \Delta x$ .

5) A house constructed over the rectangular region  $0 \leq x \leq 4$ ,  $0 \leq y \leq 6$  has a roof given by  $z = f(x, y) = 2x + y + 1$ . We wish to obtain the volume of the house.

- A. Outline the region of the house on the  $xy$  plane of your kit and place its roof.
- B. If there are two divisions in  $x$  and two divisions in  $y$  and the height of the roof in each division is to be approximated using the smallest value for each variable in each division, find  $x_1, x_2, y_1$  and  $y_2$  and use them to fill in the following table with numerical values.

Division	Length	Width	height	volume
1				
2				
3				
4				

- C. Use the values of  $x_1, x_2, y_1$  and  $y_2$  to fill in the same table below using  $x_1, x_2, y_1, y_2, \Delta x$  and  $\Delta y$  instead of numerical values. (Note, the divisions should not change between the two tables.)

Division	Length	Width	height	volume
1				
2				
3				

4				
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- D. Express the approximate volume numerically.
- E. Express the volume obtained in part D in the form  $\sum \sum (...) \Delta y \Delta x$ .
- F. Take the appropriate limits to convert the sum in part D to an integral.

- 6) A 32 km rectangular area that is populated with foxes represents the density of foxes with the function  $f$  with inputs  $x$  and  $y$  (both km) and output density ( $\frac{\text{foxes}}{\text{km}^2}$ ) as  $z = f(x, y) = xy + x$  and we wish to obtain the number of foxes that live in the area  $0 \leq x \leq 4, 0 \leq y \leq 8$ .

- A. If there are two divisions in  $x$  and four divisions in  $y$  and the density in each region is to be approximated using the middle value for each variable in each division in each division, find  $x_1, x_2, y_1, y_2, y_3$  and  $y_4$  use them to fill in the following table with numerical values.

Division	Length	Width	Density	Number of foxes
1				
2				
3				
4				
5				
6				
7				
8				

- B. Use the values of  $x_1, x_2, y_1, y_2, y_3$  and  $y_4$  to fill in the same table below using  $x_1, x_2, y_1, y_2, y_3, y_4, \Delta x$  and  $\Delta y$  instead of numerical values. (Note, the divisions should not change between the two tables.)

Division	Length	Width	Density	Number of foxes
1				
2				
3				
4				
5				
6				
7				
8				

- C. Express the approximate number of foxes numerically.
- D. Express the mass obtained in part C in the form  $\sum \sum (...) \Delta y \Delta x$ .
- E. Take the appropriate limits to convert the sum in part D to an integral.