NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

## Problem of the Week Teacher Packet

# Shipping Packages


You design boxes for the "Peculiar Package Company." This company prides itself on creating boxes in unique shapes. However, it also makes standard rectangular prism-shaped boxes.

The postal service has recently adopted a policy of only shipping packages that meet certain size requirements. The new regulations state, "The maximum combined length and girth is not to exceed 100 inches."

Your new assignment is to design a package in the shape of a rectangular prism with a square base and the largest possible volume. It must also meet the postal service requirement.

What are the dimensions of the package? Give your answer in its exact form - no decimal approximations, please.

### Answer Check

After students submit their solution, they can choose to "check" their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually get the answer) we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

*The dimensions of the package that yield the maximum volume are 50/3 in x 50/3 in x 100/3 in.*

If your answer does not match our answer,

- Did you know that girth meant the length all the way around the package, a.k.a. the perimeter of a cross section?
- Did you remember that the combined length and girth can't exceed 100?
- Did you draw a picture?
- Did you remember that the base of the box has to be square?
- If you pick a length for one of the sides, can you figure out what the other ones have to be, or can they vary?
- Can you write an expression for finding the volume given one or more of the sides?
- Do you know some ways to find the maximum or minimum value of a function?
- Did you try making a graph?
- Did you try making a table?
- Did you find the maximum volume?
- Did you check your arithmetic?
- Did you round at any point that could have given you a different answer?

If any of those ideas help you, you might revise your answer.

If your answer does match ours,

- Did you explain any calculus concepts that you used?
- If you used a guess and check strategy, did you explain how you chose the numbers you guessed?
- Did you use another method to check your work?

- Did you tell us about any "Aha!" moments you may have had?
- Could you explain your work so another student who was stuck could understand how to solve it?
- If you used a calculator, did you explain how you used it and what the calculator was doing?
- Did you make any mistakes along the way?

Revise your work if you have any ideas to add. You might answer one or more of the questions above.

## Sample Student Solutions

The solutions below represent correct solutions that we received from students when we first used this problem. For each, we've provided a short note giving a little more insight into the student's strategy and what we're on the lookout for as mentors.

**Calclueless, age 17, Derivative, Maximization, and Finding the Volume**

The dimensions of the package are as follows: Length: (50/3) inches; Width: (50/3) inches; Height (100/3) inches

*I notice that Calclueless used the derivative to find the solution. This was the most common method that students used.*

Let L be the length of the box across (height)

Let x be one side of the square base

Let V be the volume of the box

We first set up an equation that combines the length and the girth of the box. The girth of the box would be the perimeter around the box, so this value would be 4x. Since the combined length and girth cannot exceed 100, we set the equation equal to 100. The equation is:

*I wonder if they could find another method. For example, could they find a solution method that a ninth grader could follow?*

L + 4x = 100

The next step is to find the equation for the volume of the box because this is what we want to maximize. Because the box has a square base, the length and width are equal. L is the length of the box across, so this is the height. The equation for the volume is:

V = x^2 * L

Now, we solve for L and substitute into the volume equation:

L = 100 - 4x

V = x^2 * (100 - 4x)

V = 100x^2 - 4x^3

We want to maximize the volume of the box, so we need to find the derivative of V with respect to x:

dV/dx = 200x - 12x^2

Because a quadratic function has only one global extremum (at its vertex) and the squared term is negative, we know that by setting the derivative equal to 0, it will give us the maximum value of x:

0 = 200x - 12x^2

0 = 4x(50 - 3x)

x = {0 , (50/3)}

The solution x = 0 is not an extremum, so x = (50/3) is our maximum side length of the square base. We substitute this value back into our original equation to find the maximum length across:

L + 4(50/3) = 100

L = 100 - (200/3)

L = (100/3)

The girth is 4x, so the maximum girth would be (200/3) inches. Therefore, the maximum dimensions of the box would be (100/3) inches for the height, (50/3) inches for the width, and (50/3) inches for the length.

**Jason, age 17, Derivative, Maximization, and Finding the Volume**
The width is 50/3 inches and the length is 100/3 inches

The girth of an object is the distance around it, or the perimeter of its base.

This package has a square base, so the girth would equal "4 w."

Since the length and the girth can be no greater than 100 inches, you can set up an equation:

4 w + L = 100

Now solve for "w":

4 w = 100 - L

w = 25 - (L/4)

The volume of a rectangular prism is:

V = L * w * h

Since this package has a square base we can rewrite this formula to:

V = L * (w^2)

Substitute (25 - (L/4)) in for "w":

V = (L)((25 - (L/4))^2)

(FOIL) the "(25 - (L/4)) * (25 - (L/4))":

V = L * (625 - (25L/2) + (L^2)/16)

Distribute the "L" throughout the parentheses:

V = 625 L - (25(L^2))/2 + (L^3)/16

To find the largest possible volume, you have to take the derivative of the volume equation (or you could graph it on the calculator, but that would be too easy).

V' = 625 - 25 L + (3(L^2))/16

Now, you have to find the "critical values," or where the derivative is equal to zero.

0 = 625 - 25 L + (3(L^2))/16

Next, you have to use the quadratic formula to solve for "L"

```
     25 + or - (625 - (1875/4))^(1/2)
L = ----------------------------------
                 (3/8)

     25 + or - (625/4))^(1/2)
L = ----------------------------------
                 (3/8)


     25 + or - (25/2)
L = ------------------
          (3/8)

     (75/2)       (25/2)
L = -------- or --------
     (3/8)         (3/8)

L = 100 or 100/3 inches
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*I notice Jason also uses the derivative to solve for this problem.*

*I also noticed that he used the quadratic formula to solve for length, whereas others found the length by using the volume equation.*

*Even though he said it would be too easy to solve the problem with a graph, I wonder if he checked his work on the calculator.*

Since 100 inches doesn't make any sense, the length of the package must be (100/3) inches.

L = (100/3) inches

Go back and substitute "(100/3)" into the equation where you solved for "w" earlier (w = 25 - (L/4))

w = 25 - ((100/3)/4)

w = 25 - (25/3)

w = (50/3) inch

**Saul, age 17, Derivative and Second Derivative**

The dimensions of the rectangular prism, which meets the specified restrictions, and has a maximum volume is (50/3)"X(50/3)"X(100/3)".

Start with what was given:

the length and girth of this box cannot exceed 100 inches.

ohkay, so L+G=100in; since it is obviously wise to use as much of the material as is provided for this box.

next we know that the box has a square base and so:

we can label each of its sides by a single variable, just to be traditional, i will say each side has a value x.

progressing, we come upon a relationship between the girth of the box and the sides of the square. The girth must be equal to the perimeter of the box's base, or in mathematical terms, 4x.

the next logical step is to come up with an equation for the volume of the box in the problem.

Well, the box is a rectangular prism, so it must follow the equation,

V=area of the base*h(or in this case L actually) V=x^2*L

hmmm, but this equation is no good, it has to variables in it, one can't have that... so it is back to the first equation we go:

L+G=100in

but we know G=4x, so L=100-4x, great now this new representation of L can be plugged into the previously determined volume formula.

V=x^2*(100-4x)=-4x^3+100x^2

so we now have V as a function of x, and finally we can practice some actual calculus. Looks like the derivative would be handy here, because as everyone knows, when a function's derivative is zero it usually a signals a maximum or minimum of that function.

V'=-12x^2+200x; set the derivative equal to zero. (-12x+200)x=0; and so x=0 or x=(50/3), hmmm, somehow i doubt that the box will have a maximum value when the boxes area is zero, so it is probably best to chose (50/3)inches.

whoa, can't stop just yet, we haven't actually proven whether or not our x value yields a maximum or a minimum value. For the answer, let us turn to the second derivative test.

V''=-24x+200; Plug in (50/3)

*I noticed Saul used the derivative and second derivative test to solve for this problem.*

*I also notice that he works out the problem in a very logical way. He talks through the problem, which is clearly evident through his work, which helps support development in his problem solving skills.*

*I wonder if this was the exact thought process that went through his mind when completing the problem, or if there was more that he didn't include, such as other possible ways he attempted to solve the problem.*

V''=-24(50/3)+200=-200; the significance here is the sign because it means the graph's slope is decreasing, and therefore the point at which the slope equals zero(50/3) must be a maximum.

**Kris, age 17, Guess and Check**

The height of the box will be 16 2/3 inches, the width of the box will also be 16 2/3 inches, and the length of the box will be 33 1/3 inches.

This problem was not very hard but took some time to solve because of the guess and check process used. I began by writing out some equations to match the problem. I knew that the girth is the distance around something. For my equations, G=girth, L=length, H=height, and W=width. I first realized that G=2W+2H. Since L+G=100, I substituted the other equation in to find that L+2W+2H=100. Because the base had to be a square, I knew that in this case the width and height had to be equal in measure. Due to this, my equation became even simpler; L+4H=100. From here I used guess and check to solve the problem and make sure all conditions were correct for all equations. I came to my final answer of H=16 2/3 inches, W=16 2/3 inches, and L=33 1/3 inches through the guess and check process.

Below are some of the guesses which I checked.....

| Height | Width | Length | Area |
|---|---|---|---|
| 20 | 20 | 20 | 8000 |
| 15 | 15 | 40 | 9000 |
| 17 | 17 | 32 | 9248 |
| 18 | 18 | 28 | 9072 |
| 16 | 16 | 36 | 9216 |
| 16.5 | 16.5 | 34 | 9256.5 |
| 16.666 | 16.666 | 33.333 | 9258.4=Greatest |

*I noticed Kris used the Guess and Check method to solve this problem.*

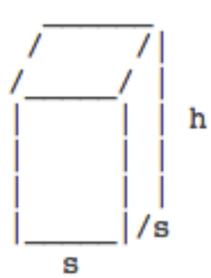
*I noticed that she still used algebraic methods though she did not solve or graph her equations. She was able to solve this problem without using calculus.*

*I wonder, if prompted, if she would be able use a graph to show she found the only possible answer.*

**Andrea, age 17, Derivative and Second Derivative**

The maximum dimensions of the package are 50/3 in, by 50/3 in, by 100/3 in.

Since we are dealing with a rectangular prism-shaped box, the figure will look like this:




With h as the height and s as the sides of the square base, the maximum combined length and girth is no greater than 100 inches. Therefore, the volume is:

V = (s^2)*h

and the combined length and girth is:

4s + h = 100

h = 100 - 4s

Notice also that s belongs to the closed interval [0, 25].

So, when setting up the problem, we substitute h in the volume with 100 - 4s in order to make the volume in terms of s:

V(s)= (s^2)*(100 - 4s)

= 100s^2 - 4s^3

*I notice Jason also uses the derivative to solve for this problem.*

*I also noticed that he used the quadratic formula to solve for length, whereas others found the length by using the volume equation.*

*Even though he said it would be too easy to solve the problem with a graph, I wonder if he checked his work on the calculator.*

Now we take the first derivative:

$V'(s)= 200s -12s^2$

$= s(200 - 12s)$

The critical points are x= 0, 50/3. Since you cannot have a dimension of 0, the only critical point left is 50/3.

Now we take the second derivative:

$V"(s)= 200 - 24s$

$V"(50/3)< 0$

We plug in the critical point into the second derivative. Notice how this will give you a negative answer, implying the presence of a maximum.

Now in addition to these calculations, we compare the values of V(s) for s = 0, 50/3, and 25.

$V(0) = 0$

$V(25) = 0$, and

$V(50/3) = 250000/9$

All of these comparisons, along with the confirmation of the 2nd derivative test, lead us to conclude that x = 50/3 gives the absolute maximum for the volume on the closed interval.

To figure out the dimensions of the package, we need to solve for h, the height.

$h = 100 - 4s$, where $s = 50/3$

$h = 100 - 4(50/3)$

$= 100/3$

Therefore, the dimensions are 50/3 in, by 50/3 in, by 100/3 in.

## Standards

If your state has adopted the Common Core State Standards, you might find the following alignments helpful.

*High School: Geometry: Geometric Measurements and Dimension:*

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

*High School: Geometry: Modeling with Geometry:*

3. Apply geometric methods to solve design problems.

*High School: Functions: Interpreting Functions:*

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity*.

*Mathematical Practices:*

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
6. Attend to precision.

## Teaching Suggestions

his problem relies on students understanding of optimization (finding maxima) as well as visualizing 3D shapes. Most of the students who attempted this problem were successful. Common mistakes in this problem were not using sound methods to find extrema (such as graphing, making an exhaustive table, or most reliably, taking the derivative and finding zeroes), not being sure what girth is or calculating it incorrectly and failing to find the single largest value.

Some students did not think to differentiate to find the local maxima. Even students of calculus may not know when to apply their knowledge of finding local maxima and minima in a contextual problem. Students who haven't yet studied calculus simply don't know those techniques. However this problem is still accessible to those students. Exhaustive guess and check and looking at the graph are both effective, and can be efficient, methods to solve this problem. Calculus students can be encouraged to look back at their initial solution to find the connections to calculus techniques they may know.

Many students tried using guess and check methods to solve this problem, but did not explain their work clearly. Using a guess and check method resulted in many of them not finding the largest possible value for the volume. A thorough guess and check method involves explaining how you know you're exactly on, or close enough to, a local maximum, and ideally involves explaining whether there could be other maxima in other "neighborhoods".

Prompting students to attempt multiple methods can help them learn how different calculus ideas are related. When having students attempt this problem, first help them understand exactly what is being asked of them. They may need to identify words they don't know and think of strategies for finding their meaning (for example, Max had to Google "girth" to solve this PoW). Then, students might be asked to draw diagrams to help them visualize the problem. Looking at other students' diagrams can help students learn what makes visualization particularly useful, and improve their own visualization skills. Drawing a really good diagram can often times trigger the students to reach that "aha!" moment. It is vital that students go into the problem solving process with a solid understanding of what is being asked of them, and move forward with the confidence that they know what they are doing. Although this problem is designed to involve calculus methods that students learn in calculus class, it could be interesting to have students attempt to solve it using a guess and check method and/or graphing prior to teaching the lesson on derivatives and double derivatives. They may notice some patterns in their tables and graphs that connect to derivatives. Be sure to encourage students to talk out their problem solving processes with others.

We hope these packets are useful in helping you make the most of Trig & Calculus Problems of the Week from the Library. Please let us know if you have ideas for making them more useful.

*https://www.nctm.org/contact/*