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PK 12

Putting Technology in Its Place

Using four mathematical tasks as models, the authors discuss how carefully designed tasks and orchestrated discussions can reveal and shape students' mathematical practices.

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Should calculators and other technology be allowed in the mathematics classroom? Such questions have initiated a lot of debate, just because of the abundance of sophisticated technology. Many calculators and software programs can quickly and accurately perform the mathematical computations, procedures, and algorithms that comprise traditional K–12 mathematics content, including, but not limited to, graphing equations and multivariable functions, factoring polynomials (of even a wider range than taught in any algebra course), simplifying expressions, performing geometric constructions, computing derivatives, performing definite and indefinite integrals, and solving systems of equations. Consequently, some teachers forbid the use of technology, being concerned that it will displace mathematics learning by circumventing the learning of these skills, computations, and procedures. Others have embraced its potential to entertain students or capture their interest—being used, perhaps, with little benefit for student understanding and development.

What is effective use of technology? We assert that technology is used effectively when it enables students to engage in authentic mathematical activity. The most powerful and transferable aspects of mathematics are not its procedures, processes, and algorithms but rather the *mathematical practices* that give rise to them. These practices include the values, habits of mind, and ways of knowing that enable mathematicians to solve new problems, devise new strategies, and establish their validity. These include, but are not limited to, the eight Standards for Mathematical Practice (SMP) of the Common Core State Standards (NGA Center and CCSSO 2010) and the five Process Standards of the National Council of Teachers of Mathematics (NCTM 2000). Technology is used effectively in the classroom when it enables students to engage in such practices, when it is the means to an end, serving as a catalyst or scaffold for mathematical activity rather than being the goal itself (Boaler 2016). This can be done by selecting or choosing mathematical tasks that necessitate the use of particular mathematical practices, practices that might be inaccessible without technology.

In this article, we discuss how technology, in conjunction with carefully designed tasks and orchestrated discussions, has the potential to both reveal our students' mathematical practices and to provide opportunities to shape those practices. View video 1 for a short introduction about our methods. We examine four specific mathematical practices that tend to be associated with problem-solving situations, namely—

1. considering multiple examples before generalizing;
2. seeking out diverse or extreme examples;
3. looking systematically at cases; and
4. attending to definitions.

For each of these practices, we give an example of a task and suitable technology and discuss ways of facilitating the development of the practice. Online applets have been created to illustrate for the reader what students might create or see while exploring each task. Whether to provide applets like these to students warrants serious pedagogical thought and planning because doing so can influence which mathematical practices students employ, how they engage in those practices, and the mathematical knowledge and ideas they develop and access.

Video 1 Methods of Using Technology

Watch the full video online.

CONSIDERING MULTIPLE EXAMPLES BEFORE GENERALIZING

Mathematics is the study of abstract ideas, ideal objects, and their relationships. Most mathematical concepts did not originate as abstract ideas but were abstracted from work with concrete problems and representations. Professional mathematicians and expert problem solvers know that understanding and insights are often hard-earned and come by generating, organizing, and studying many examples. Because of this, experts often hold off on generalizing and conjecturing, seeking first to understand through experience, exploration, and experimentation; they *consider multiple examples before generalizing*.

Relevant to many algebra courses, task 1 affords students an opportunity to explore graphical transformations (e.g., shifts, compressions, and reflections) resulting from alterations to a function's defining equation. Graphing technology, such as graphing calculators or software like

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Desmos (https://www.desmos.com/calculator/vkt0tihuea) or GeoGebra (https://ggbm.at/zyh6uxav), provides students with a way of creating accurate graphs quickly, removing the time-consuming graphing process this might look like. This enables students to consider many examples, which is essential for the pattern recognition and generalization needed for this task.

Task 1 Exploring Transformation of Functions

Consider the function

$$f(x) = x^5 + 2x^4 - 7x^3 - 8x^2 + 12x$$

How do the graphs of each of these modified versions of the function f compare with the original? Make some conjectures about their relationships to the graph of f:

$$f(x) + c$$
$$f(x + c)$$
$$f(cx)$$
$$c \bullet f(x)$$

Will your conjectures be true for *all* functions? Why or why not?

Considering multiple examples is critical to this task because doing so acts as a window into students' generalization tendencies. Students who have not developed the propensity to consider multiple cases tend to give up or generalize from only one or two examples. For instance, they use a few simple integers for *c*, such as 1, 2, 3, resulting in overly simplistic, incorrect, or unclear relationships within the task.

Activities like this also afford teachers opportunities to foster student acquisition of this practice by encouraging persistence, example generation, and reflection on those examples. During student exploration, example generation can be encouraged using such questions as these:

- What examples have you tried so far?
- How do your examples compare or differ?
- How can you change the graphs?

Persistence and a healthy skepticism are promoted by asking such questions as these:

- How do you know it always works?
- Did you consider other examples?

Teachers can also encourage students to create a paper trail of their examples, allowing students to compare and contrast the results.

Reflective discussion about students' mathematical practices will provide further encouragement. Teachers can have students share their examples by asking, "What did you *do* to come up with that conjecture?" They can then shift the focus of the class to the consideration of multiple examples by asking, "Why was it important for us to look at several examples? What might we have missed if we had not done so?" Such overt discussion about investigating multiple examples can help students become aware of this strategy and develop the disposition to avoid overgeneralizing.

SEEKING OUT DIVERSE OR EXTREME EXAMPLES

Many mathematical ideas arise only by studying a diverse set of examples. Mathematicians recognize (but students may not) that objects defined by some common properties may not share all properties (Belnap and Parrott 2013). Failing to explore diversity during inductive work can result in incorrect generalizations, assumptions, or abstracted properties. Although many textbooks and teachers intentionally model this practice by presenting diverse examples, because it can be subtle, students might not recognize and incorporate this practice without explicit intervention (Alcock 2004).

Task 2 provides students with a particularly challenging situation, namely, exploring the definition of an unconventional angle bisector quadrilateral (ABQ). Students must exercise their creativity and problem-solving skills to both understand the definition and to uncover these quadrilaterals' properties. Robust geometry environments (e.g., Geometer's Sketchpad® or GeoGebra) allow students to construct and manipulate dynamic models of the objects. These models enable students to observe real-time changes in the object's properties and uncover covariational relationships. Students can also uncover extreme and unforeseen cases, such as when the resulting quadrilateral degenerates to a point.

Task 2 Exploring Angle Bisector Quadrilaterals

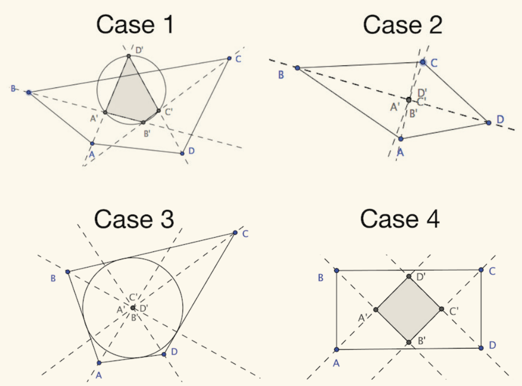
Consider a generic quadrilateral *ABCD*. The angle bisector quadrilateral of *ABCD* is a quadrilateral $A'B'C'D'$, where A', B', C', and D' are each the intersection of the angle bisectors of the corresponding and subsequent vertices (e.g., A' is the intersection of the bisectors of $\angle DAB$ and $\angle ABC$). Your task is to explore the relationships between these two quadrilaterals and make several conjectures on the basis of your observations.

Seeking out unlike or extreme examples is important to this task because of the diversity of objects and mathematical situations this definition encompasses—one such construction can be seen at https://www.geogebra.org/geometry/nxmhwb9p (see figure 1). For example, students might observe that the angle bisector quadrilateral $A'B'C'D'$ of quadrilateral *ABCD*—

- has an orientation opposite that of quadrilateral *ABCD* (see cases 1 and 4 in figure 1);
- is degenerate (a single point) if quadrilateral *ABCD* has reflectional symmetry along a diagonal (i.e., it is a kite) or can have a circle inscribed in it (see cases 2 and 3 in figure 1); and
- is a square if and only if quadrilateral *ABCD* is a nonsquare rectangle (see case 4 in figure 1.).

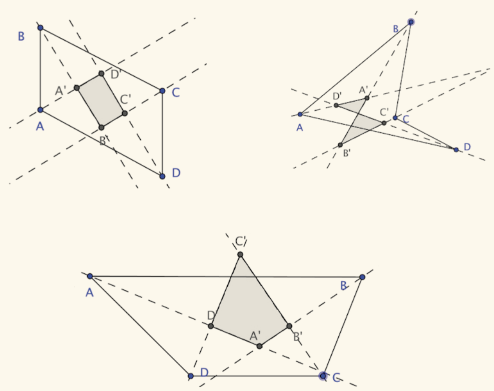
Teachers can encourage working with diverse examples by addressing students' motivation, lateral thinking, and discipline-based observational skills. First, students must develop the *propensity* to consider and search for extreme cases. Teachers can motivate this practice by fostering a healthy skepticism toward observations by treating students' unjustified answers as conjectures, challenging (rather than sanctioning) them, and proposing false solutions. For example, after writing students' ideas on the board, you might say, "These are interesting conjectures. Can you find a counterexample to any of them? Have you looked at any unusual or different quadrilaterals, like parallelograms, concave quadrilaterals, or trapezoids?" (See figure 2.) Also, instead of affirming solutions, play devil's advocate by making such comments as, "I have another solution I want you to consider: I think that every ABQ is a rhombus. Do you believe my statement? Why or why not?" After students have had time to explore, then ask, "Which conjectures do you still believe are true? Does anyone have any counterexamples

Fig. 1



These examples show diverse and extreme cases that might arise during task 2.

Fig. 2



These additional extreme examples might arise from task 2.

to share?” Then have students share counterexamples or ask them, “Why do you believe that conjecture to be true? Let’s try to prove it. What would we need to show?”

Second, students must develop an ability to *diversify their thinking*. Many students have an unconscious tendency toward homogenous, aesthetically pleasing, and simplistic examples. Often students’ “random” creations have very nice properties; it takes conscious effort to find unusual and extreme cases. Teachers can make the search for diversity and radical examples the explicit goal of classroom activity. When introducing an object through definition, have students concoct a set of three or four examples of the object, each different from the others in some fundamental way. Students take turns adding examples to the board, each time explaining how their example meets the definition yet is different from those already given. For example, with the ABQ definition, students might show examples like those in figures 1 and 2. Students begin to see ways that objects can differ without losing their defining properties.

Third, students must learn to *recognize* extreme, unusual, or interesting situations when they encounter them. Some students tend to overlook diversity, not recognizing special cases or circumstances, even when they see them occur (Belnap and Parrott 2013). This ability may be more of an art, developed subtly from experiences. If so, students need to see, share, and discuss extreme situations, broadening their ideas about what interesting or unusual circumstances might arise. Students also need encounters when things go wrong—seeing conjectures or definitions fail and apparent patterns break down.

LOOKING SYSTEMATICALLY AT CASES

Mathematical work is rarely random in nature. Organizing, systematizing, and examining cases are powerful and can be rewarding mathematical practices. Through them, mathematicians are able to strategically and intentionally search out deep and complex mathematical relationships from failed or inconclusive situations. In this way, mathematicians *look for and make use of structure* (SMP 7) so to create and reveal patterns, uncover special situations, and build understanding.

Situated in the congruence theorems of Euclidean geometry, task 3 provides an opportunity for students to investigate the complexity behind why the side-side-angle (SSA) relationship fails to guarantee triangle congruence. Because there is no guarantee that SSA will result in congruence, students must find ways to systematically explore and break down the situation into different cases, some of which *do* and some of which *do not* guarantee congruence. Dynamic geometry environments (e.g., GeoGebra or Sketchpad) help by providing precise, visual, and dynamic models that can be manipulated in various ways. Students can search for and test (both visually and through targeted constructions) such cases. (See https://www.geogebra.org/classic/pvqdmhbt for an applet that allows students to explore adjusting the angle and sides of their triangles.)

The tendency to organize one’s investigation can be powerful for this task because a variety of possible conjectures exist, each with different conditions, which can be uncovered through systematic approaches. Students who do not systematically consider cases may struggle significantly with this task, often finding only a single conjecture.

Task 3 Side-Side-Angle Uniqueness

Side-side-angle is not a congruence theorem because it does not work in all cases.

1. Give an example of two triangles that satisfy the side-side-angle relationship but are not congruent. (A picture is sufficient here.)
2. In certain situations, side-side-angle *will* yield two congruent triangles. Spend 10 to 15 minutes experimenting with various triangles. Write some conjectures describing conditions under which SSA *will* guarantee that the two triangles are congruent. Give some examples to support your conjectures.

Teachers can help students develop this practice by creating and discussing situations in which examining cases is important or useful, such as when something different occurs, or there is a need to simplify situations. Teachers can ask questions to encourage organization or systematic approaches:

- Did you try different types of triangles?
- How might you organize all of these examples that you have?

- What happens if you just change one of the three measurements?

Discussion around these questions might lead to a chart, such as in table 1.

Teachers can encourage different organizational schemes (see table 1) and the systematic exploration of each case. Debriefing can focus on similarities between the cases and ways in which the cases may be condensed. To encourage the practice, students should share, compare, and discuss how they systematically approached the problem and the ways that they organized their work.

UNPACKING MATHEMATICAL DEFINITIONS

Definitions are important and powerful mathematical tools; they provide precise conditions for classifying objects, which empowers us to make mathematical arguments and communicate effectively. Mathematicians understand that to attain this precision, definitions are written in a condensed, compact, and concise manner. Consequently, developing an understanding of and intuition about what is defined (i.e., unpacking the definition) requires inductive work and exploration, such as generating, comparing, and contrasting examples and nonexamples.

Within the context of descriptive statistics, task 4 gives students an opportunity to unpack the mathematical definitions of *mean*, *median*, and *mean absolute deviation* (MAD) by requiring students to coordinate statistics and data set creation. By removing cumbersome computations from students' cognitive load, technology like calculators and spreadsheets can allow students to focus on experimentation and observation, for example, see https://ggbm.at/tqqnwr9m. Students are then free to focus on how changes in the underlying data set affect these statistics, thereby helping them develop an understanding of what they measure, how to interpret them, how they relate to a data distribution, and what their biases are.

Task 4 Measures of Central Tendency

Create the following three data sets, each of which is composed of exactly five integers that are between 1 and 10.

1. Data Set A: A set with a mean of 5 and a median of 4
2. Data Set B: A set with the largest possible mean absolute deviation (MAD)
3. Data Set C: A set with the largest possible difference between the mean and the median

Unpacking mathematical definitions is critical to this task (and task 2, Exploring Angle Bisector Quadrilaterals) because it requires students to dig deeper into the underlying ideas that are captured by the definitions but not explicitly stated. For example, although *mean, median,* and *MAD* are typically defined as formulas on a data set, which are procedural in nature for students, the questions posed cannot be addressed or answered by simple calculation. Students must develop an understanding of the attribute that each statistic measures by learning how the data set produces the specific statistic. Limiting the

Table 1 One Organizational Scheme for the Side-Side-Angle Uniqueness Task

	Angle ($\angle A$)		
Sides	$\angle A$ is acute	$\angle A$ is right	$\angle A$ is obtuse
Side adjacent to $\angle A$ is shorter than the side opposite of $\angle A$			
Side adjacent to $\angle A$ is congruent to the side opposite of $\angle A$			
Side adjacent to $\angle A$ is longer than the side opposite of $\angle A$			

number of data points pushes students to be more strategic in their selections and encourages them to think carefully about the meaning of each measure and how to change it.

Unpacking mathematical definitions requires mathematical skills, such as the ability to generate examples and nonexamples, the ability to read mathematical statements, and logical reasoning. Teachers can help students acquire the relevant skills by spending time unpacking new definitions with their students. One way to do this is to have students generate examples (and nonexamples) of the definition—note that exploring nonexamples provides contrasting cases that further develop conceptual understanding, distinguishing properties the objects do and do *not* have. Teachers can also ask students to restate definitions in their own terms or to draw pictorial representations of the definition when appropriate. Additionally, teachers can help students experience definitions by directing students to them frequently. For example, when a student asks if she computed the mean correctly, instead of sanctioning or correcting the student's response, the teacher could direct the student to the definition by saying, "Let's see. What does the definition say?" When students ask questions that can be settled by definitions, directing them to the definitions is compelling because it sends two important messages: (1) *Mathematical conventions and reasoning* are the authority in determining correctness, not the teacher, and (2) *they* can understand mathematics and can determine for themselves when they are correct.

Finally, understanding the power and purpose of definitions need not be subtle. Teachers can explicitly talk about their purpose and how they empower us to make decisions and formulate arguments. See Szydlik, Parrott, and Belnap (2016) for more about discussing definitions with students.

CONCLUSION

Technology is interesting and exciting. It can improve our lives in many ways, but it can also be an ineffective distraction or an obstacle to learning. Along with technology, we *must* attend to the big question of how we can use technology to facilitate and enhance rather than detract from and impede student learning.

The key, of course, lies in *students'* actions and thinking. When technology replaces mathematical thinking, it impedes student learning. Conversely, when technology gives students access to authentic mathematical activity and practices, it enhances education because it provides both a window into our students' mathematical practices and opportunities to foster and shape our students' views, tendencies, and abilities. By focusing on the development of mathematical practices, we can use technology to empower our students to reason mathematically.

Mathematical practices can be fostered through strategic technology use, but they do not spontaneously develop from mere technological presence; they must be nurtured. As teachers, helping students establish these practices should be an explicit focus of our instruction. This may require student engagement in strategic tasks, making them the focus of discussion, teacher modeling, and intervention. As teachers present opportunities for students to meaningfully engage with technology, *students* gain a chance to develop their mathematical practices *and we* can see and influence those practices.

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